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Quasi random numbers in stochastic finite element analysis

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Abstract – A non-intrusive stochastic finite-element method is proposed for uncertainty propagation through mechanical systems with uncertain input described by random variables. A polynomial chaos expansion (PCE) of the random response is used. Each PCE coefficient is cast as a multi-dimensional integral when using a projection scheme. Common simulation schemes, e.g. Monte Carlo Sampling (MCS) or Latin Hypercube Sampling (LHS), may be used to estimate these integrals, at a low convergence rate though. As an alternative, quasi-Monte Carlo (QMC) methods, which make use of quasi-random sequences, are proposed to provide rapidly converging estimates. The Sobol' sequence is more specifically used in this paper. The accuracy of the QMC approach is illustrated by the case study of a truss structure with random member properties (Young's modulus and cross section) and random loading. It is shown that QMC outperforms MCS and LHS techniques for moment, sensitivity and reliability analyses.

Key words: Uncertainty propagation / stochastic finite-elements / non intrusive spectral approach / polynomial chaos expansion / quasi-random numbers / structural reliability

Résumé – Utilisation des nombres quasi-aléatoires dans la méthode des éléments-finis stochastiques. On s'intéresse dans cet article à une méthode aux éléments-finis stochastiques non intrusive pour la propagation d'incertitudes à travers des systèmes mécaniques dont les paramètres incertains sont représentés par des variables aléatoires. On représente la réponse aléatoire du système sur la base dite du chaos polynomial. Chaque coefficient de ce développement est exprimé sous la forme d'une intégrale multi-dimensionnelle au moyen d'une méthode de projection. Des techniques de simulation classiques, telles que les simulations de Monte Carlo (MCS) ou les tirages par hypercube latin (LHS), peuvent être employées pour estimer ces intégrales. Cependant, la vitesse de convergence des estimateurs associés est lente. De manière alternative, on propose d'utiliser les méthodes quasi-Monte Carlo (QMC), basées sur les suites à discrépance faible ou suites quasi-aléatoires (e.g. la suite de Sobol') pour obtenir des estimateurs à convergence rapide. La précision de la méthode QMC est illustrée sur l'exemple d'un treillis dont les propriétés des barres (module d'Young et section droite) ainsi que les sollicitations sont aléatoires. On montre la supériorité de QMC sur MCS et LHS pour les analyses de distribution, de sensibilité et de fiabilité.

Mots clés : Propagation d'incertitudes / éléments-finis stochastiques / approche non intrusive / chaos polynomial / nombres quasi-aléatoires / fiabilité des structures

1 Introduction

Computer simulations are nowadays commonly used in structural engineering to accurately model the behaviour of complex systems. Most of them are deterministic and thus provide relevant information as long as the input data is well known, which is seldom the case in reality. Probabilistic methods allow to take into account such uncertainty by modeling the input parameters by random variables, thus leading to a random response. The latter may be efficiently represented using *polynomial chaos expansions* [1, 2]. The coefficients of this representation can be expressed as a multidimensional integral using a non intrusive projection scheme [3–9]. In this paper, the use of *quasi-random numbers* [10, 11] is proposed to provide rapidly converging estimates of the PCE coefficients, as suggested in [5]. As shown in the sequel, this approach

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allows to efficiently compute various quantities of interest, e.g. statistical moments, sensitivity indices and probabilities of failure, at a lower computational cost compared to other non intrusive computational schemes.

The basic formulation of polynomial chaos expansions is recalled in Section 2. Then the computation of the PCE coefficients is detailed in Section 3 using a non intrusive projection method. The use of quasi-random numbers is addressed in this section. Then the various byproducts of a polynomial chaos expansion are presented in Section 4, namely the computation of the response PDF, the statistical moments and specific sensitivity measures called *Sobol' indices*. The use of a PC expansion in the context of reliability analysis is then addressed. Finally an application example dealing with a truss structure involving 10 independent random variables is given in Section 5.

2 Polynomial chaos expansion

Consider a physical system whose uncertain parameters are modeled by *independent* random variables $(X_1, ..., X_M)$ gathered into a random vector $\boldsymbol{X}$. The system behaviour is described by a deterministic function f which can be analytical or more generally algorithmic (e.g. a finite element model). The propagation of the uncertainty in the input through the model function f leads to a random response of the system. This random response denoted by $Y = f(\boldsymbol{X})$ is supposed to be scalar throughout the paper without loss of generality (indeed, the computational schemes presented in the sequel are applicable componentwise in case of vectorial response quantities). Provided Y has a finite variance, it can be expressed in an orthonormal polynomial basis as follows:

$$Y = f(\boldsymbol{X}) = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^M} a_{\boldsymbol{\alpha}} \psi_{\boldsymbol{\alpha}}(\boldsymbol{X}) \tag{1}$$

This expansion is referred to as *finite-dimensional polynomial chaos expansion* (PCE) [2]. The $a_{\boldsymbol{\alpha}}$'s are unknown deterministic coefficients and the $\psi_{\boldsymbol{\alpha}}$'s are multivariate polynomials which are orthonormal with respect to the joint probability density function (PDF) $p_{\boldsymbol{X}}$ of the input random vector $\boldsymbol{X}$, i.e. $E[\psi_{\boldsymbol{\alpha}}(\boldsymbol{X})\psi_{\boldsymbol{\beta}}(\boldsymbol{X})] = 1$ if $\boldsymbol{\alpha} = \boldsymbol{\beta}$ and 0 otherwise. As the random variables are assumed to be independent, their joint PDF reads:

$$p_{\boldsymbol{X}}(\boldsymbol{x}) = p_{X_1}(x_1) \times \cdots \times p_{X_M}(x_M) \tag{2}$$

and the $\psi_{\boldsymbol{\alpha}}$'s can be constructed as the tensor products of M unidimensional polynomials $P^{(i)}_{\alpha_i}$ that form an orthonormal family with respect to the marginal PDF p_{X_i}:

$$\psi_{\boldsymbol{\alpha}}(\boldsymbol{X}) = \prod_{i=1}^{M} P^{(i)}_{\alpha_i}(X_i) \tag{3}$$

with $E[P^{(i)}_k(X_i)\, P^{(i)}_l(X_i)] = 1$ if $k = l$ and 0 otherwise, for all $(k, l) \in \mathbb{N}^2$. For computational purposes, the PC expansion in Equation (1) is truncated so that the maximal degree of the polynomials does not exceed p:

$$f(\boldsymbol{X}) \simeq \sum_{|\boldsymbol{\alpha}| \leq p} a_{\boldsymbol{\alpha}}\, \psi_{\boldsymbol{\alpha}}(\boldsymbol{X}) \tag{4}$$

where $|\boldsymbol{\alpha}| = \sum \alpha_i$. The number of terms in the above summation is given by $P = \binom{M+p}{p}$. An analytical representation of the random response $\boldsymbol{Y}$ can thus be obtained by computing the coefficients $a_{\boldsymbol{\alpha}}$, which is the scope of the following section.

3 Non-intrusive computation of the PCE coefficients

The PCE coefficients can be computed using a *non-intrusive projection approach* [3–5], which exploits the orthogonality of the PCE basis. Indeed, by premultiplying Equation (1) by $\psi_{\boldsymbol{\alpha}}(\boldsymbol{X})$ and by taking the expectation of the product, one gets the exact expression of each coefficient $a_{\boldsymbol{\alpha}}$:

$$a_{\boldsymbol{\alpha}} = E[f(\boldsymbol{X})\psi_{\boldsymbol{\alpha}}(\boldsymbol{X})] \tag{5}$$

which can be rewritten as:

$$a_{\boldsymbol{\alpha}} = \int_{\mathcal{D} \subset \mathbb{R}^M} f(\boldsymbol{x})\, \psi_{\boldsymbol{\alpha}}(\boldsymbol{x})\, p_{\boldsymbol{X}}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \tag{6}$$

where $\mathcal{D}$ denotes the support of $\boldsymbol{X}$. The multidimensional integral (6) can be computed using several simulation schemes, as described in the sequel.

3.1 Monte Carlo sampling

Monte Carlo Sampling (MCS) [12] is based on the generation of independent pseudo-random numbers $(\boldsymbol{x}^{(1)}, ..., \boldsymbol{x}^{(N)})$ according to the input joint PDF $p_{\boldsymbol{X}}$. The expectation in (5) is estimated by the empirical mean:

$$\hat{a}^{N,\mathrm{MC}}_{\alpha} = \frac{1}{N} \sum_{i=1}^{N} f\left(\boldsymbol{x}^{(i)}\right)\, \psi_j(\boldsymbol{x}^{(i)}) \tag{7}$$

The mean-square error of estimation reads:

$$E\left[\left(a_\alpha - \hat{a}^{N,\mathrm{MC}}_{\alpha}\right)^2\right] = \frac{\mathrm{Var}[f(\boldsymbol{X})\psi_\alpha(\boldsymbol{X})]}{N} \tag{8}$$

This shows the familiar convergence rate $\mathcal{O}(N^{-1/2})$ associated with MCS.

3.2 Latin hypercube sampling

The Latin Hypercube Sampling (LHS) method [13] aims at generating pseudo-random numbers with a better uniformity over $I_M = [0, 1]^M$ compared to MCS. The domain is divided into N equiprobable intervals or *stratas*, leading to a partition of I_M in equiprobable subsets. Let us consider M independent uniform random variables $(U_1, ..., U_M)$ over $[0,1]$. N realizations of each U_i are randomly generated by selecting one value in each strata. The N realizations of U_1 are randomly paired without replacement with the N realizations of U_2. The resulting N pairs are then randomly combined with the N realizations of U_3, and so on until a set of N M-dimensional samples is formed. The latter is finally transformed into a set of pseudo-random numbers that are distributed according to $p_{\boldsymbol{X}}$. LHS has been commonly used in a stochastic analysis framework for computing the PCE coefficients, see e.g. [3].

3.3 Quasi-Monte Carlo sampling

The convergence rate associated with the pseudo-random schemes mentioned above can often be increased by using highly uniform sets of numbers over I_M, which are called *low discrepancy sequences* or *quasi-random numbers* [10, 11]. The use of such sequences for computing the PCE coefficients has been suggested in [5]. Upon introducing a mapping from $\mathcal{D}$ to I_M, the integral in Equation (6) reads:

$$a_{\boldsymbol{\alpha}} \;=\; \int_{I_M} f\big(T^{-1}(\boldsymbol{u})\big)\; \psi_{\boldsymbol{\alpha}}\big(T^{-1}(\boldsymbol{u})\big)\; \mathrm{d}\boldsymbol{u} \tag{9}$$

where $T : \boldsymbol{X} \longmapsto \boldsymbol{U}$ is the isoprobabilistic transform of each component of $\boldsymbol{X}$ into a uniform random variable $\mathcal{U}[0,1]$. Let $(\boldsymbol{u}^1, \ldots, \boldsymbol{u}^N)$ be a set of N quasi-random numbers. The *quasi-Monte Carlo* (QMC) estimate of $a_{\boldsymbol{\alpha}}$ is given by:

$$\hat{a}_\alpha^{N,\mathrm{QMC}} \;=\; \frac{1}{N}\sum_{i=1}^{N} f\big(T^{-1}(\boldsymbol{u}^{(i)})\big)\; \psi_{\boldsymbol{\alpha}}\big(T^{-1}(\boldsymbol{u}^{(i)})\big) \tag{10}$$

The Koksma-Hlawka inequality [11] provides an upper bound of the absolute estimation error:

$$\left|a_\alpha - \hat{a}_\alpha^{N,\mathrm{QMC}}\right| \;\leq\; V\big(f\psi_{\boldsymbol{\alpha}}\big) D_N\left(\boldsymbol{u}^{(1)}, \ldots, \boldsymbol{u}^{(N)}\right) \tag{11}$$

where $V(f\psi_{\boldsymbol{\alpha}})$ denotes the so-called *total variation* of $f\psi_{\boldsymbol{\alpha}}$, which depends on the mixed derivatives of $f\psi_{\boldsymbol{\alpha}}$, and D_N represents the *star discrepancy* of the quasi-random sample, which measures its uniformity. In this paper, the use of the *Sobol' quasi-random sequence*, for which D_N converges at the rate $\mathcal{O}(N^{-1}\ln^M(N))$, is proposed.

The unidimensional Sobol' sequence is a particular low discrepancy sequence defined as follows. Consider the binary expansion of a natural integer $n \in \mathbb{N}$:

$$n \equiv (q_m \cdots q_0)_2 \tag{12}$$

The nth term is the Sobol' sequence reads:

$$u^{(n)} \;=\; \sum_{i=0}^{m} \frac{q_i}{2^{i+1}} \tag{13}$$

Figure 1 shows the space-filling process of $[0,1]$ using this technique. The M-dimensional sequences are built by pairing M permutations of the unidimensional sequences. Figure 2 shows the space-filling process of $[0,1]^2$ using a two-dimensional Sobol' sequence, compared to MCS and LHS, from which the better uniformity of the former is obvious.

4 Postprocessing of the polynomial chaos expansion

4.1 Statistical moments

The statistical moments of the response PCE can be analytically derived from its coefficients. In particular, the mean and the variance respectively read:

$$\mu_Y \;=\; a_{\boldsymbol{0}} \tag{14}$$

$$\sigma_{Y,P}^2 \;=\; \sum_{0<|\boldsymbol{\alpha}|\leq p} a_{\boldsymbol{\alpha}}^2 \tag{15}$$

where $(\cdot)_{,P}$ recalls that only $P = \binom{M+p}{p}$ terms have been retained in Equation (4). The *skewness coefficient* of the response is defined as:

$$\delta_Y \;=\; \frac{1}{\sigma_Y^3}\, E\left[\big(Y - \mu_Y\big)^3\right] \tag{16}$$

Its PCE-based approximation reads:

$$\delta_{Y,P} \;=\; \frac{1}{\sigma_{Y,P}^3} \sum_{0<|\boldsymbol{\alpha}|,|\boldsymbol{\beta}|,|\boldsymbol{\gamma}|\leq p} a_{\boldsymbol{\alpha}} a_{\boldsymbol{\beta}} a_{\boldsymbol{\gamma}} \times E\left[\psi_{\boldsymbol{\alpha}}(\boldsymbol{X})\psi_{\boldsymbol{\beta}}(\boldsymbol{X})\psi_{\boldsymbol{\gamma}}(\boldsymbol{X})\right] \tag{17}$$

Note that the expectations in the above equation are zero for many sets of multi-indices $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ and can thus be efficiently stored in a sparse structure. If the $\psi_{\boldsymbol{\alpha}}$'s are products of Hermite polynomials, these expectations can be computed analytically [14]. Otherwise a quadrature scheme can be used. In the same way, the *kurtosis coefficient* of the response is given by:

$$\kappa_Y \;=\; \frac{1}{\sigma_Y^4}\, E\left[\big(Y - \mu_Y\big)^4\right] \tag{18}$$

and is approximated as follows:

$$\kappa_{Y,P} \;=\; \frac{1}{\sigma_{Y,P}^4} \sum_{0<|\boldsymbol{\alpha}|,|\boldsymbol{\beta}|,|\boldsymbol{\gamma}|,|\boldsymbol{\delta}|\leq p} a_{\boldsymbol{\alpha}}\, a_{\boldsymbol{\beta}}\, a_{\boldsymbol{\gamma}} a_{\boldsymbol{\delta}} \times E\left[\psi_{\boldsymbol{\alpha}}(\boldsymbol{X})\psi_{\boldsymbol{\beta}}(\boldsymbol{X})\psi_{\boldsymbol{\gamma}}(\boldsymbol{X})\psi_{\boldsymbol{\delta}}(\boldsymbol{X})\right] \tag{19}$$

4.2 Sensitivity indices

Variance-based methods of sensitivity analysis aim at quantifying the relative importance of each input parameter in the response variance. The *Sobol' sensitivity indices* [15, 16] have been proposed for this purpose:

$$S_i \;=\; \frac{\mathrm{Var}\left[E[Y|X_i]\right]}{\sigma_Y^2} \tag{20}$$

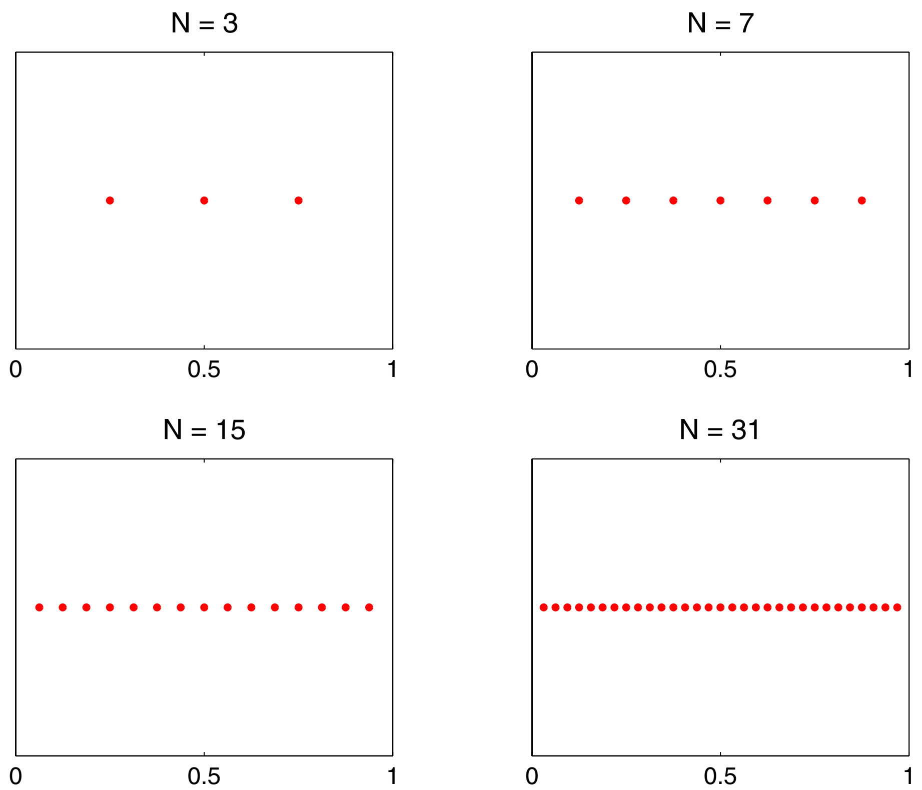


Fig. 1. Space-filling process of $[0, 1]$ using a unidimensional Sobol' sequence.

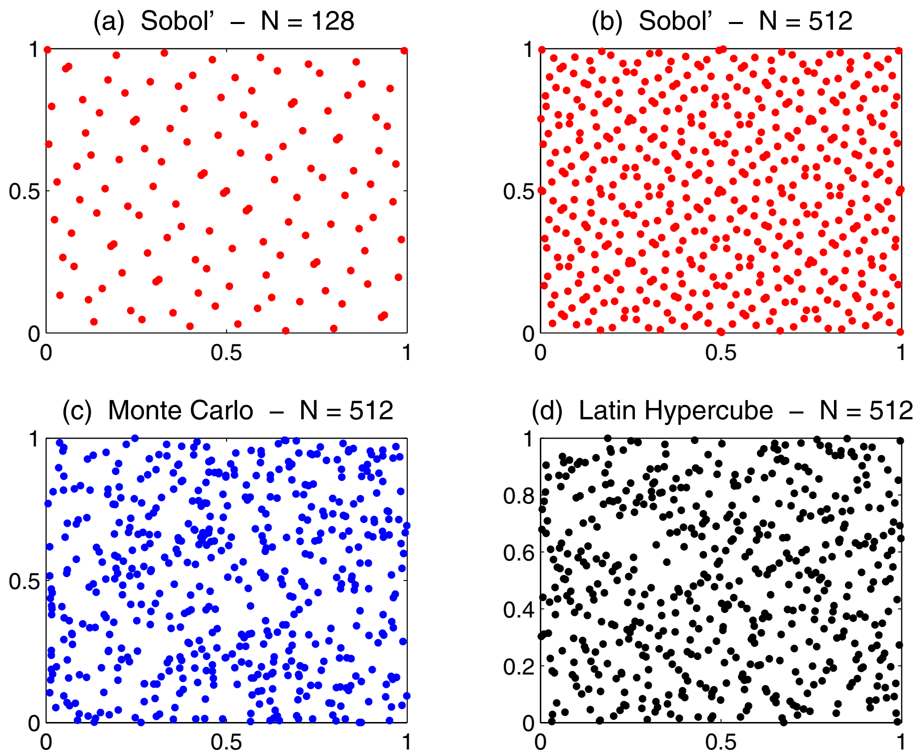


Fig. 2. Space-filling process of $[0, 1]^2$ using a two-dimensional Sobol' sequence, compared to MCS and LHS.

S_i is referred to as the *first order sensitivity index* associated to X_i. This measure can be extended to each subset $\{i_1, \ldots, i_s\}$ of input random variables to quantify the interaction effects:

$$S_{i_1,\ldots,i_s} = \frac{\mathrm{Var}\left[E[Y|X_{i_1}, \ldots, X_{i_s}]\right]}{\sigma_Y^2} \tag{21}$$

As the number of such indices dramatically increases with the number of input random variables, one often computes instead the so-called *total Sobol' sensitivity indices* of each input X_i, which are defined as the sum of all the Sobol' indices involving i:

$$S_{T_i} = \sum_{\boldsymbol{u} \subset \{1,\ldots,M\} \backslash \{i\}} S_{\{i\} \cup \boldsymbol{u}} \tag{22}$$

As shown in [17], the Sobol' indices can be analytically computed from the coefficients of the PCE in Equation (1). Deriving Equation (15) conditionally to each X_i, one gets the following PCE-based estimates of the partial variances $\mathrm{Var}\left[E[Y|X_i]\right]$:

$$D_{i,P} = \sum_{\boldsymbol{\alpha} \in I_i^P} a_{\boldsymbol{\alpha}}^2 \tag{23}$$

where I_i^P is the set of multi-indices $\boldsymbol{\alpha}$ which appear in Equation (1) and such that only the index i is non zero:

$$I_i^P = \Big\{ \boldsymbol{\alpha} \in \mathbb{N}^M \;:\; (|\boldsymbol{\alpha}| < p) \\ \wedge \left(\alpha_k = 0 \Leftrightarrow k \neq i \quad \forall\, k = 1, \ldots, M \right) \Big\} \tag{24}$$

Consequently, the PCE-based estimates of the first order Sobol' indices read:

$$SU_{i,P} = \frac{1}{\sigma_{Y,P}^2} \sum_{\boldsymbol{\alpha} \in I_i^P} a_{\boldsymbol{\alpha}}^2 \tag{25}$$

The extension to the PCE-based estimates of the general Sobol' indices is straightforward:

$$SU_{(i_1,\ldots,i_s),P} = \frac{1}{\sigma_{Y,P}^2} \sum_{\boldsymbol{\alpha} \in I_{i_1,\ldots,i_s}^P} a_{\boldsymbol{\alpha}}^2 \tag{26}$$

where:

$$I_{i_1,\ldots,i_s}^P = \Big\{ \boldsymbol{\alpha} \in \mathbb{N}^M \;:\; (|\boldsymbol{\alpha}| < p) \\ \wedge \left(\alpha_k = 0 \Leftrightarrow k \notin (i_1, \ldots, i_s) \quad \forall\, k = 1, \ldots, M \right) \Big\} \tag{27}$$

Hence the PCE-based total Sobol' indices:

$$SU_{T_i,P} = \sum_{\boldsymbol{u} \subset \{1,\ldots,M\} \backslash \{i\}} S_{\{i\} \cup \boldsymbol{u},P} \tag{28}$$

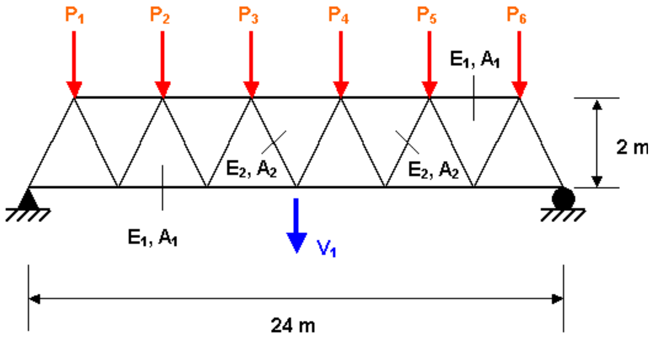


Fig. 3. Truss structure with 23 members.

Table 1. Truss example, input random variables.

Variable	Distribution	Mean	Standard deviation
E_1, E_2 (Pa)	Lognormal	2.10×10^{11}	2.10×10^{10}
A_1 (m^2)	Lognormal	2.0×10^{-3}	2.0×10^{-4}
A_2 (m^2)	Lognormal	$1.0{\times}10^{-3}$	$1.0{\times}10^{-4}$
P_1–P_6 (N)	Gumbel	$5.0{\times}10^4$	$7.5{\times}10^3$

5 Application example

5.1 Problem statement

Let us consider the truss structure sketched in Figure 3, already considered in [18]. Ten independent input random variables are considered, whose distribution, mean and standard deviation are reported in Table 1.

The deflection at midspan v is regarded as the model response and approximated by a second-order ($p = 2$) PCE made of normalized multivariate Hermite polynomials. In this respect, it is necessary to transform the input random vector $\boldsymbol{Z} = \{E_1, E_2, A_1, A_2, P_1, \ldots, P_6\}^T$ into a standardized Gaussian vector:

$$\xi_i = P_{Z_i}^{-1}\left(\Phi(\,Z_i)\right), \; \forall\, i \in \{1, \ldots, 10\} \tag{29}$$

where Φ denotes the standard normal CDF. This leads to:

$$v^{\mathrm{PCE}}(\boldsymbol{\xi}) = \sum_{0 \leq |\boldsymbol{\alpha}| \leq p} a_{\boldsymbol{\alpha}} \psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}), \quad P = \binom{10+2}{2} = 66 \tag{30}$$

where P denotes the number of terms.

5.2 Moment analysis

The four first statistical moments of the response v are computed from the PCE coefficients, which are estimated using the MCS, LHS and QMC schemes. On the one hand, reference values have been computed using crude Monte Carlo simulation of the problem (1 000 000 samples are used). On the other hand, estimates of the PCE coefficients are obtained using $N = 10\,000$ samples and the moments of the response are post-processed using Equations (14), (15), (17), (19). The resulting response moments are reported in Table 2 together with the reference values.

Table 2. Truss example – estimates of the four first statistical moments.

	Reference	MCS	LHS	QMC
Number of FE runs	1 000 000	10 000	10 000	10 000
μ_v	–0.0794	–0.0792	–0.0794	–0.0794
σ_v	0.0111	0.0120	0.0124	0.0112
δ_v	–0.4920	–0.5605	–0.2052	–0.4959
γ_v	3.4555	3.9667	3.2944	3.3676

Accurate estimates of the mean value are obtained when using LHS and QMC, whereas the MCS scheme also yields a rather insignificant relative error $\varepsilon = 0.3\%$ with respect to the reference value. However, QMC provides the best estimate of the standard deviation ($\varepsilon = 0.9\%$) whereas relative errors of 8.1% and 11.7% are associated with MCS and LHS respectively. Moreover, the QMC estimate of the skewness coefficient is much closer ($\varepsilon = 0.8\%$) to the reference value than those provided by MCS and LHS ($\varepsilon = 13.9\%$ and 58.3% respectively). Finally, QMC also provides the best estimate of the response kurtosis coefficient, yielding a relative error $\varepsilon = 2.5\%$ versus $\varepsilon = 14.8\%$ and $\varepsilon = 4.7\%$ using MCS and LHS respectively.

The probability density function of the maximal deflection can be estimated by the various methods. The reference solution corresponds again to 1 000 000 runs of the finite element model. The PCE-based PDF corresponds to 1 000 000 samples of the PCE obtained by the various methods (where 10 000 finite-element runs have been used in each case to compute these PC coefficients). In all cases, a kernel representation of the sample set of maximal deflection is used [19] (Fig. 4). It appears that the PDF obtained from the QMC approach is much closer to the reference solution (i.e. almost identical) than that derived from the MCS and LHS computation, both in the central part and in the tails (see the plot with logarithmic scale to the right in Fig. 4).

5.3 Sensitivity analysis

The total Sobol' indices are computed from the PCE according to equation (28). The estimates associated to the use of MCS, LHS and QMC for computing the PCE coefficients are reported in Table 3 together with reference values. The reference values are computed from standard Monte Carlo formulae [15]:

$$D_{\sim i}^{\rm MC} = \frac{1}{N}\sum_{m=1}^{N} f\left(\boldsymbol{x}^{(m)}\right) f\left(\boldsymbol{x}_{\sim i,1}^{(m)}, x_{i,2}^{(m)}\right) - \mu_v,$$

$$S_{T_i}^{\rm MC} = 1 - \frac{D_{\sim i}^{\rm MC}}{\sigma_v^2} \quad (31)$$

In this formula, the subscripts in $\boldsymbol{x}_{\cdot,1}$ and $\boldsymbol{x}_{\cdot,2}$ indicate that two different Monte Carlo samples are generated and mixed. Moreover, subscript $_{\sim i}$ corresponds to vectors in which the ith component has been discarded:

$$\boldsymbol{x}_{\sim i}^{(m)} = \left(x_1^{(m)}, \ldots, x_{i-1}^{(m)}, x_{i+1}^{(m)}, \ldots, x_M^{(m)} \right) \quad (32)$$

The empirical mean in equation (31) is computed using $N = 500\,000$ samples, which actually leads to perform $N \times (M+1) = 5\,500\,000$ deterministic finite element runs [17] to compute the whole set of total Sobol' indices.

It can be concluded from the Sobol' indices that the variability of the deflection v is much more sensitive to the variables E_1 and A_1, than E_2 and A_2. This makes sense from a physical point of view since the properties of the horizontal bars are more influential on the displacement at midspan than the oblique ones. It can been also observed that the Sobol' indices associated with E_1 and A_1 (resp. E_2 and A_2) are similar. This is due to the fact that these variables have the same type of PDF and coefficient of variation, and that the displacement v only depends on them through the products E_1A_1 and E_2A_2. Finally, the Sobol' indices reflect the symmetry of the problem, giving similar importances to the loads that are symmetrically applied (e.g. P_3 and P_4). Greater sensitivity indices are logically attributed to the forces that are close to the midspan than those located at the ends.

Such physically meaningful indices are obtained using $N = 10\,000$ QMC samples. In contrast, MCS and LHS can provide unreliable results, as overestimating the importance E_2 for instance. In all cases, the QMC scheme yields more accurate estimates than MCS and LHS.

5.4 Reliability analysis

The serviceability of the structure with respect to an admissible maximal deflection is studied. The associated limit state function reads:

$$g(\boldsymbol{z}) = v_{\max} - |v(\boldsymbol{z})| \leq 0, \; v_{\max} = 0.11\,\mathrm{m} \quad (33)$$

The reference value of the probability of failure has been obtained by crude Monte Carlo simulation:

$$P_{\rm f}^{\rm MC} = \frac{N_{\rm fail}}{N} \quad (34)$$

where $N = 1\,000\,000$ samples and $N_{\rm fail}$ is the number of samples corresponding to a negative value of the limit state function Equation (33). The result is $P_{\rm f}^{\rm MC} = 8.7 \times 10^{-3}$, and the coefficient of variation of the underlying estimate is 1.1%. The corresponding *reliability index* is given by $\beta^{\rm MC} = -\Phi^{-1}(P_{\rm f}^{\rm MC}) \simeq 2.38$.

On the other hand, once the PCE coefficients have been obtained by MCS, LHS or QMC using 10 000 finite element runs, an approximate limit state function is built:

$$g^{\rm PCE}(\boldsymbol{z}(\boldsymbol{\xi})) = v_{\max} - |v^{\rm PCE}(\boldsymbol{\xi})| \quad (35)$$

The probability of failure is then computed from Equation (35) (which is a polynomial function almost costless to evaluate) using 1 000 000 Monte Carlo samples.

Results are reported in Table 4. QMC provides the most accurate estimate with a relative error on β of 0.8%, whereas relative errors of 8.8% and 3.8% are respectively associated with MCS and LHS.

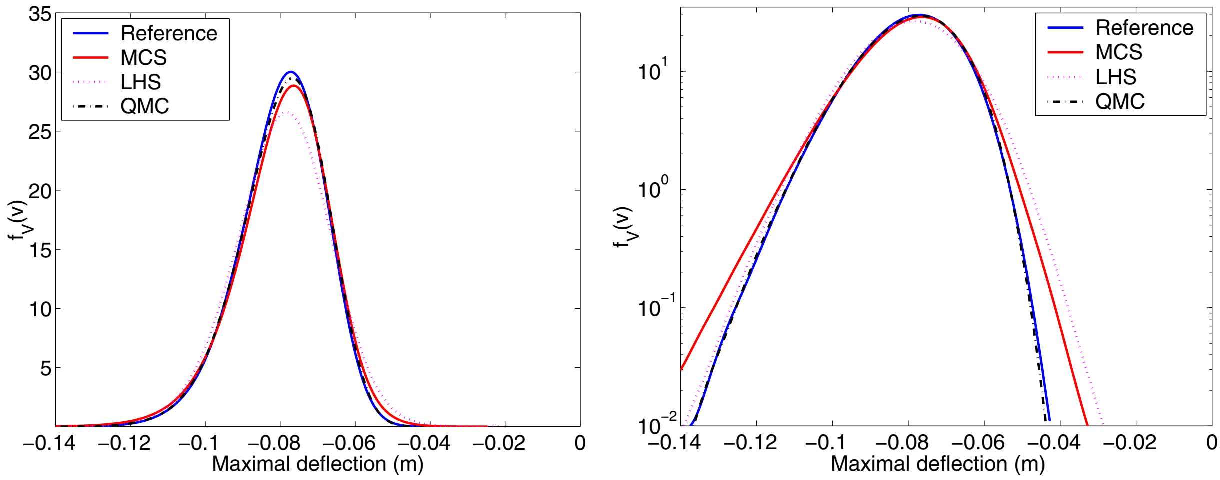


Fig. 4. Truss example – probability density function of the maximal deflection (left: linear scale; right: logarithmic scale).

Table 3. Truss example – estimates of the total Sobol' indices.

Variable	Reference	MCS	LHS	QMC
Number of FE runs	5 500 000	10 000	10 000	10 000
A_1	0.388	0.320	0.344	0.366
E_1	0.367	0.356	0.331	0.373
P_3	0.075	0.067	0.095	0.077
P_4	0.079	0.124	0.080	0.077
P_5	0.035	0.086	0.068	0.046
P_2	0.031	0.079	0.067	0.039
A_2	0.014	0.074	0.052	0.014
E_2	0.010	0.088	0.115	0.013
P_6	0.005	0.067	0.013	0.014
P_1	0.004	0.037	0.063	0.005

Table 4. Truss example – estimates of the probability of failure and the reliability index.

	Reference	MCS	LHS	QMC
Number of FE runs	1 000 000	10 000	10 000	10 000
P_{f}	8.7×10^{-3} (CV = 1.1%)	1.5×10^{-2}	1.1×10^{-2}	9.1×10^{-3}
β	2.38	2.17	2.29	2.36
Relative error on β	–	8.8%	3.8%	0.8%

Finally, a parametric study is carried out to assess the accuracy of the QMC estimates of P_{f} and β when varying the threshold $v_{\max}$. The QMC estimates are obtained using $N = 10\,000$ samples. Results are reported in Table 5 together with the reference values.

As expected, the estimation error increases with the threshold value, i.e. when the probability of failure decreases, with a relative error on β varying from 2.9% to 4.7%. This indicates that a second order PC expansion is not sufficiently accurate to describe the far tails of the response PDF. Better reliability results are thus expected using a third order PCE [8]. However, as the total variation of the monomials $\psi_{\boldsymbol{\alpha}}$'s increases with their degree $|\boldsymbol{\alpha}|$, so does the QMC estimation error according

Table 5. Truss example – reliability results obtained by projection using QMC (parametric study).

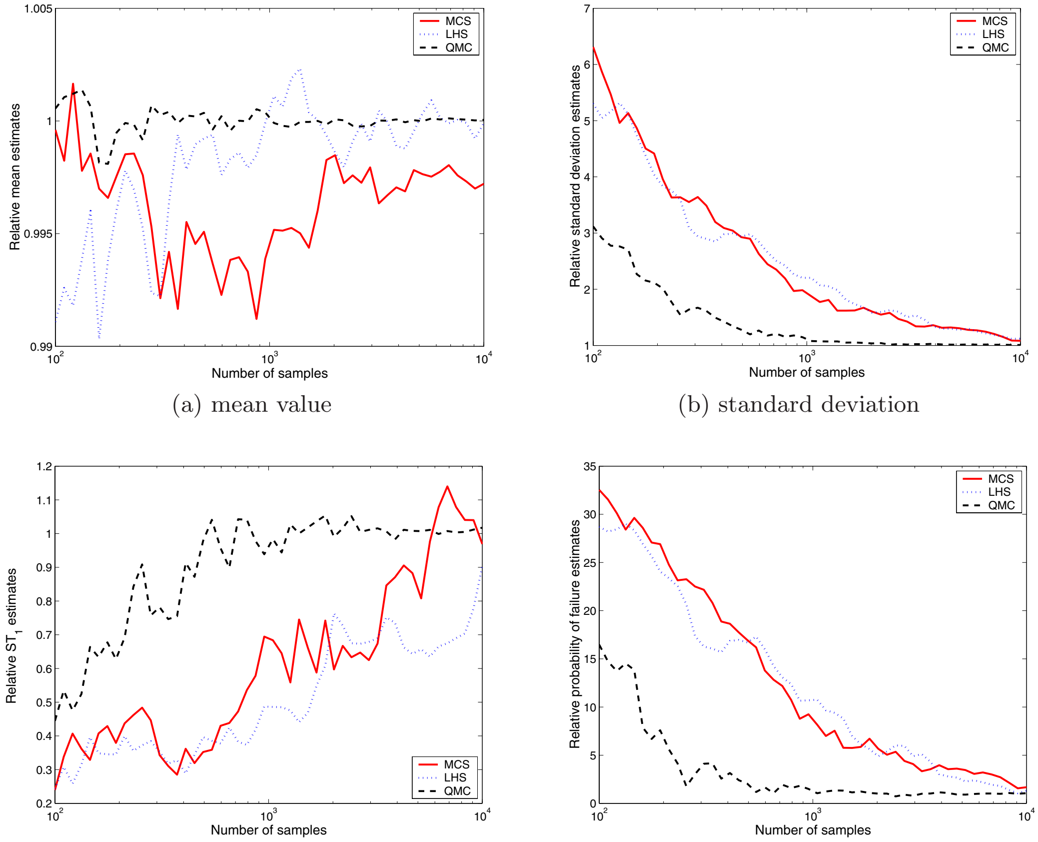
Threshold (cm)	Ref. solution		QMC Simulation	
	P_{f}	β	P_{f}	β
10	4.0×10^{-2}	1.75	4.4×10^{-2}	1.70
11	8.7×10^{-3}	2.38	9.1×10^{-3}	2.36
12	1.5×10^{-3}	2.97	1.4×10^{-3}	2.98
14	3.5×10^{-5}	3.98	2.1×10^{-5}	4.09
16	6.0×10^{-7}	4.85	1.8×10^{-7}	5.08

to Equation (11). A number of samples N greater than 10 000 would probably be required to provide accurate estimates of the 3rd order PC coefficients.

The convergence rates associated to the various methods are plotted in Figure 5. In each subfigure, the ratio of the quantity of interest with respect to its reference value is plotted as a function of the number of samples N used to compute the PCE coefficients. Figures 5a–d show the convergence of the mean value, standard deviation, total Sobol' index S_{T_1} and probability of failure (for $v_{\max} = 11$ cm) respectively. It can be observed in all four cases that QMC converges more rapidly than MCS and LHS. Rather accurate results are obtained by QMC from 1000 sample whereas the convergence of MCS and LHS for 10 000 samples seems not to be attained.

6 Conclusion

In the context of non intrusive stochastic finite elements, the quasi-Monte Carlo method has been proposed to provide rapidly converging estimates of the response polynomial chaos expansion coefficients. Post-processing techniques for an efficient computation of the statistical moments of the response, the Sobol' sensitivity indices and probabilities of failure are reported, together with a method to plot accurately the response probability density function.



(a) mean value (b) standard deviation

(c) total Sobol' index S_{T_1} (d) probability of failure

Fig. 5. Convergence rates of the simulation-based estimators.

The QMC method is based on the generation of deterministic quasi-random sequences which ensure a better space-filling of the unit hypercube than the pseudo-random numbers used in classical Monte Carlo schemes. Three specific quasi-random sequences, namely the Halton, Faure and Sobol' sequences, had been already successfully used in [20] to analyse the sensitivity of models whose input random variables were uniformly distributed. From this analysis it was concluded that the Sobol' sequence was the most efficient choice.

In the present paper, the Sobol' sequence is compared to MCS and LHS in an application example dealing with a finite element model of truss and non uniform input random variables. The QMC scheme overperformed the other methods, providing reliable estimates of the response statistical moments, sensitivity indices and probability of failure, using about 1000 samples, i.e. 1000 deterministic finite-element runs. In contrast, the MCS and LHS methods seem not to have converged even for $N = 10\,000$ samples. Consequently, this method seems to be an efficient alternative to Monte Carlo when using a simulation scheme to evaluate the PCE coefficients. In the future, it would be interesting to assess the performance of QMC in higher dimensions (e.g. $M \geq 20$) and in the case of non-linear models.

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