Mechanics & Industry **17**, 107 (2016)

DOI: 10.1051/meca/2015051
www.mechanics-industry.org

Mechanics & Industry

Free vibration analysis of homogeneous and FGM skew plates resting on variable Winkler-Pasternak elastic foundation

Mohamad Javad Ketabdari, Alireza Allahverdi, Sayad Boreyri[a] and Manouchehr Fadavie Ardestani

Department of Marine Technology, Amirkabir University of Technology, Tehran, Iran

Received 25 February 2015, Accepted 24 June 2015

Abstract – In this research study, free vibration of homogeneous and functionally graded skew plates resting on Winkler-Pasternak elastic foundation is investigated. The elastic foundation is assumed to be a combination of Winkler and Pasternak elastic support with linearly or parabolically variable stiffness coefficients along the directions. Plate skewness is obtained by using a transformation from Cartesian coordinate to oblique coordinate system. The energy of the functionally graded skew plate and the elastic foundation is derived, and the natural frequency of the plate is calculated by the Rayleigh-Ritz method. The results are compared with available results in the literature, showing an excellent agreement. Furthermore, a parametric study is carried out to thoroughly investigate the effects of different boundary conditions, skew angles, inhomogeneity factors, and variable elastic foundation stiffness on the free vibration of skew plates.

Key words: Free vibration / skew plate / FGM plate / Winkler-Pasternak elastic foundation / Rayleigh-Ritz method

1 Introduction

Composite materials are manufactured based on different industrial needs to optimize the response to external loads and reduce the residual and thermal stresses at desired regions of structures. Functionally graded materials (FGMs) were first introduced by the Japanese researchers in 1984 [1]. They are relatively new composites with spatially continuous variation of mechanical properties along one or more directions. This is achieved by gradually changing the composition of the constituent materials, usually ceramics and metals, so that dealing with interfacial stress concentrations can be avoided.

Skew plates are one of the important elements in civil, aerospace and marine industries. In spite of the mathematical difficulties involved in their study, they have found a wide range of application in modern structures such as parallelogram slabs in buildings and bridges, swept wings of aircrafts and ship hulls. Therefore, many studies have been focused on these types of plates. One of the earliest researches was carried out by Kaul and Cadambe [2] to find the fundamental frequencies of skew plates using an energy-based method. Since then a number of other researches tried to find better solutions for such structures. In 2000, Wang et al. [3] used Ritz method to find vibration frequencies of skew laminated sandwich plates. In 2003, Woo et al. [4] studied the free vibration of such plates. In 2009, Zhao [5], used kp-Ritz method for studying free vibration of rectangular and skew FG plates.

Elastic foundations may shift the natural frequencies of structures. Free vibration of thin rectangular plates resting on Winkler-Pasternak elastic foundation was studied by Civalek using differential quadrature method in 2007 [6]. Akhavan et al. [7] found the exact solution for rectangular Mindlin plates resting on Winkler-Pasternak foundation in 2009. Atmane et al. [8] used a new shear deformation theory for FGM rectangular plates resting on elastic Winkler-Pasternak foundation, in 2010. More recently in 2013, Joodaky et al. [9], used an extended Kantorovich method to study the deflection of a thin skew plate resting on Winkler elastic foundation. Since then, a number of researches have been performed to thoroughly investigate the vibration characteristics of such structures. According to this literature survey and best knowledge of the authors, not a single research has been carried out on free vibration of skew homogeneous or FGM plates on Winkler-Pasternak foundation, specifically with variable foundation characteristics which are more practical in industry. The main purpose of this study is to thoroughly investigate free vibration of such structures.

[a] Corresponding author: sayyad.boreyri@aut.ac.ir

Nomenclature

$B_k,\ C_k,\ \varphi_k,\ \psi_k$	Polynomial orthogonal functions
$D,\ E$	Flexural rigidity and modulus of elasticity of the plate, respectively
$\bar{k}_{\mathrm{w}},\ \bar{k}_{\mathrm{p}}$	Winkler and Pasternak non-dimensional coefficients, respectively
$P_{\mathrm{c}}, P_{\mathrm{m}}$	Any mechanical properties of the ceramic or metal plate
T^*_{pl}	Kinetic energy of the plate
$U_{\mathrm{pl}},\ U_{\mathrm{W}},\ U_{\mathrm{P}}$	Potential energy of the plate, Winkler and Pasternak foundation, respectively
$\varepsilon\eta$	Transformed axes of x, y in oblique coordinate system
θ	Skew angle
ω, β	Frequency and normalized frequency of the plate
$\bar{\rho}$	Density of the plate
$z_{\mathrm{np}},\ z_{\mathrm{mp}}$	Neutral and mid plane location, respectively
Δ_z	Distance between z_{np} and z_{mp}

Therefore, study of skew homogeneous and FGM plates resting on variable elastic foundation is investigated in this paper. In section two, the theory and method of solution are introduced. This section mainly deals with the derivation of formulation to calculate the energy of FGM skew plates resting on Winkler-Pasternak foundations. Thereafter, the results are validated and fully discussed in section three, and finally a conclusion on the analysis of the vibration of such structures is drawn by the authors in Section 4.

2 Theory and methodology

2.1 FGM materials

The FGM plate used in this research is a plate made of ceramic and metal. Material properties of the plate vary through the thickness from pure metal to pure ceramic smoothly and continuously. Change in material properties can be modeled based on the power law distribution, the exponential distribution [10], the Mori-Tanaka scheme [11, 12] and functionally graded materials with porosities [13]. In this study, the power law distribution is employed as in the form of:

$$P\left(z\right) = \left(P_{\mathrm{c}} - P_{\mathrm{m}}\right)\left(\frac{2z+h}{2h}\right)^n + P_{\mathrm{m}};\ \frac{-h}{2} < z < \frac{h}{2} \quad (1)$$

where h is the thickness, and indexes c and m refer to ceramic and metal, respectively. $P\left(z\right)$ is the inhomogeneity function and can be either $E\left(z\right)$, elasticity modulus or $\rho\left(z\right)$, density of the plate. It can be seen that elasticity modulus and density are not constant, as they are functions of plate thickness. It is also seen that for $n = 0$, pure ceramic, and for $n = \infty$, pure metal is obtained.

It should be mentioned that the neutral plane location, z_{np} is different from the mid plane location, z_{mp} for FG plates [10, 14]. It is located at $z_{\mathrm{np}} = z_{\mathrm{mp}} + \Delta_z$. Here, Δ_z, the distance of neutral plane from the middle plane, can be calculated using Equation (2) wherein, z must be measured from the mid plane [15]

$$\Delta_z = \frac{\int_{-h/2}^{+h/2} zP\left(z\right)\mathrm{d}z}{\int_{-h/2}^{+h/2} P\left(z\right)\mathrm{d}z} \quad (2)$$

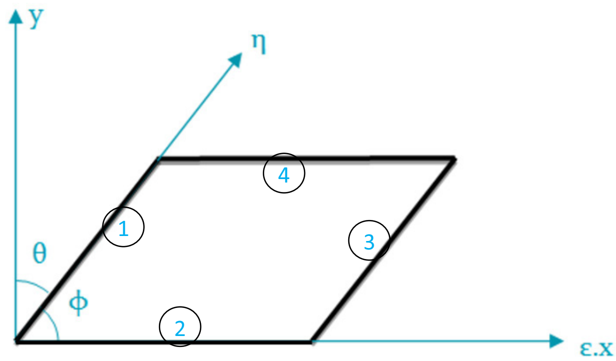

Fig. 1. Oblique and Cartesian coordinate systems.

2.2 Coordinate system

In order to analyze the structure, Cartesian coordinate system must be transformed into oblique coordinate system of Figure 1 using Equation (3) [9, 16]. It should be noted that numbers in the circles in Figure 1 refer to the sequence of boundary conditions in this study. A skew SCCC plate means the one with a simple support condition at edge 1, and clamped condition at the remaining edges, or a skew CCSS plate has clamped condition at edges 1 and 2, and simple support condition at edges 3 and 4.

$$\begin{cases} \varepsilon = x - y\tan\theta \\ \eta = \dfrac{y}{\cos\theta} \end{cases} \quad (3)$$

Consequently, the operator ∇^2 in Cartesian coordinate system is rewritten in the suitable form for oblique coordinate system as Equation (4)

$$\bar{\nabla}^2 = \frac{1}{\cos\theta}\left(\frac{\partial^2}{\partial\varepsilon^2} - 2\sin\theta\frac{\partial^2}{\partial\varepsilon\partial\eta} + \frac{\partial^2}{\partial\eta^2}\right) \quad (4)$$

2.3 Energy of the skew plate

Potential and kinetic energy of the system can be calculated using Rayleigh-Ritz method in order to derive vibration frequencies. The potential energy, U_0, and kinetic energy, T_0, of a rectangular plate can be found using Equation (5):

$$U_0 = \frac{1}{2}\iiint\left(\frac{Ez^2}{1-\upsilon^2}\right)\left\{\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}\right)^2 +2\left(1-\nu\right)\left[\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2-\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}\right]\right\}\mathrm{d}x\mathrm{d}y\mathrm{d}z, \quad (5)$$

$$T_0 = \frac{1}{2}\iiint\rho\left(\frac{\partial W}{\partial t}\right)^2\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Introducing Equations (3) and (4) into Equation (5), the maximum potential and kinetic energy of a skew FGM plate can be achieved as:

$$U_{\mathrm{pl}} = \frac{1}{2}\frac{ab}{a^4\cos^3(\theta)}\int_{-h/2}^{+h/2}\frac{E(z)z^2}{(1-\upsilon^2)}\mathrm{d}z\int_0^1\int_0^1\left\{\left(\frac{\partial^2 W}{\partial\varepsilon^2}\right)^2 +\left(\frac{a}{b}\right)^4\left(\frac{\partial^2 W}{\partial\eta^2}\right)^2+4\left(\frac{a}{b}\right)^2\sin^2\theta\left(\frac{\partial^2 W}{\partial\varepsilon\partial\eta}\right)^2 +2\left(\frac{a}{b}\right)^2\frac{\partial^2 W}{\partial\varepsilon^2}\frac{\partial^2 W}{\partial\eta^2} -4\left(\frac{a}{b}\right)\sin(\theta)\frac{\partial^2 W}{\partial\varepsilon\partial\eta}\left[\frac{\partial^2 W}{\partial\varepsilon^2}+\left(\frac{a}{b}\right)^2\frac{\partial^2 W}{\partial\eta^2}\right] -2(1-\nu)\left(\frac{a}{b}\right)^2\cos^2(\theta)\left(\frac{\partial^2 W}{\partial\varepsilon^2}\frac{\partial^2 W}{\partial\eta^2} -\left(\frac{\partial^2 W}{\partial\varepsilon\partial\eta}\right)^2\right)\right\}\mathrm{d}\varepsilon\mathrm{d}\eta, \quad (6)$$

$$T_{\mathrm{pl}} = \frac{1}{2}ab\cos\theta\int_{-\frac{h}{2}}^{+\frac{h}{2}}\rho(z)\,\mathrm{d}z\iint\left(\frac{\partial W}{\partial t}\right)^2\mathrm{d}\varepsilon\mathrm{d}\eta \quad (7)$$

2.4 Energy of the elastic foundation

The energy of the elastic foundation includes two parts: the energy due to Winkler-type foundation, Equation (8), and Pasternak-type foundation, Equation (9)

$$U_{\mathrm{W}} = \frac{1}{2}abD\cos\theta\iint\bar{k}_{\mathrm{w}}w^2\mathrm{d}\varepsilon\mathrm{d}\eta \quad (8)$$

$$U_{\mathrm{P}} = \frac{abD}{2\cos(\theta)}\iint\bar{k}_{\mathrm{p}}\left[\left(1+\left(\frac{\mathrm{a}^2}{\mathrm{b}^2}-1\right)\sin^2(\theta)\right)\left(\frac{\partial w}{\partial\varepsilon}\right)^2 +\frac{\mathrm{a}^2}{\mathrm{b}^2}\left(\frac{\partial w}{\partial\eta}\right)^2-2\frac{\mathrm{a}^2}{\mathrm{b}^2}\sin\theta\left(\frac{\partial w}{\partial\varepsilon}\frac{\partial w}{\partial\eta}\right)\right]\mathrm{d}\varepsilon\mathrm{d}\eta \quad (9)$$

$$\bar{k}_{\mathrm{w}} = \frac{k_w a^2 b^2}{D},\quad \bar{k}_{\mathrm{p}} = \frac{k_p ab}{D}$$

where U_{W}, U_{P} are potential energy due to Winkler and Pasternak foundations, and $\bar{k}_{\mathrm{w}}$, $\bar{k}_{\mathrm{p}}$ are Winkler and Pasternak non-dimensional coefficients, respectively. The total structure is depicted in Figure 2. It should be noted that $\bar{k}_{\mathrm{w}} = \bar{k}_{\mathrm{w}}(\varepsilon,\eta)$, and $\bar{k}_{\mathrm{p}} = \bar{k}_{\mathrm{p}}(\varepsilon,\eta)$, which means that the foundation stiffness is variable along the directions. The potential, U, and kinetic energy of the system can be calculated using Equations (10) and (7) respectively.

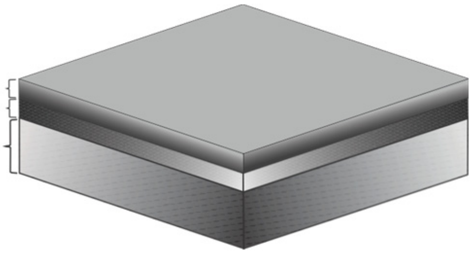

Fig. 2. An FGM skew plate on Winkler-Pasternak type foundation.

$$U = U_{\mathrm{pl}} + U_{\mathrm{W}} + U_{\mathrm{P}} \quad (10)$$

2.5 Rayleigh-Ritz formulation

For a harmonic oscillation, the kinetic energy of Equation (7) can be simplified as Equation (11). Therefore, natural frequencies can be found from Equation (12) under the condition that $w = w(\varepsilon,\eta)$ is defined.

$$T_{\mathrm{pl}}^* = \omega^2 \quad (11)$$

$$\omega^2 = \frac{U}{T^*} \quad (12)$$

To define $w(\varepsilon,\eta)$, a set of polynomial functions has been used in this study. These functions along ε-direction are in the form of Equation (13). Similar functions can be selected along η-direction, and therefore, two-dimensional orthogonal functions can be produced.

$$\varphi_1(\varepsilon) = (\varepsilon - B_1)\varphi_0(\varepsilon)$$
$$\varphi_k(\varepsilon) = (\varepsilon - B_k)\varphi_{k-1}(\varepsilon) - C_k\varphi_{k-2}(\varepsilon)$$
$$B_k = \int_a^b \varepsilon\psi(\varepsilon)\varphi_{k-1}^2(\varepsilon)\,\mathrm{d}x \Big/ \int_a^b \psi(\varepsilon)\varphi_{k-1}^2(\varepsilon)\,\mathrm{d}\varepsilon \quad (13)$$
$$C_k = \int_a^b \varepsilon\psi(\varepsilon)\varphi_{k-1}(\varepsilon)\varphi_{k-2}(\varepsilon)\,\mathrm{d}\varepsilon / \int_a^b \psi(\varepsilon)\varphi_{k-2}^2(\varepsilon)\,\mathrm{d}\varepsilon$$

Here, $\psi(\varepsilon)$ is the weight function and considered to be constant 1. B_k and C_k are some coefficients. Orthogonal functions, $\varphi_k(\varepsilon)$ should satisfy the following conditions:

$$\int_a^b \varepsilon\psi(\varepsilon)\varphi_k(\varepsilon)\varphi_l(\varepsilon)\,\mathrm{d}\varepsilon = \begin{cases} 0 & if\ k \neq l \\ a_{kl} & if\ k = l \end{cases} \quad (14)$$

The function $\varphi_0(\varepsilon)$ should be chosen according to the boundary conditions. Further information is available in references [17,18]. Substituting these equations into Equation (12) gives the natural frequencies of the skew FGM plate resting on Winkler-Pasternak foundation.

Table 1. 1st five frequency parameters of the homogeneous skew plate.

θ	method	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	% error of the 5th mode
				CCCC			
0	Present	35.9885	73.4129	73.4121	108.2574	131.7788	0.14
	[19]	35.9878	73.4048	73.4048	108.2679	131.5988	
15	Present	38.200	72.914	82.645	109.563	139.355	0.28
	[20]	38.187	72.896	82.618	109.56	138.97	
30	Present	46.092	81.602	105.199	119.303	165.970	0.59
	[20]	46.089	81.601	105.17	119.25	164.98	
45	Present	65.688	106.505	148.613	157.762	199.491	1.3
	[20]	65.643	106.49	148.31	157.23	196.77	
				SSSS			
0	Present	19.73922	49.3772	49.3772	78.9992	98.90416	0.21
	[20]	19.73921	49.34802	49.3480	78.95684	98.69608	
15	Present	20.880	48.206	56.142	79.097	104.355	0.34
	[20]	20.868	48.205	56.107	79.043	104.00	
30	Present	25.066	52.639	72.152	84.017	124.086	1.04
	[20]	24.899	52.638	71.711	83.829	122.82	
45	Present	35.269	66.295	101.120	109.815	143.197	1.7
	[20]	34.755	66.277	100.25	107.01	140.80	

3 Results and discussion

3.1 Model validation

In order to validate the model, free vibration of the homogeneous skew plates has been investigated and the non-dimensional natural frequencies have been compared to available results in the literature considering no elastic foundation, that is $\bar{k}_{\mathrm{w}} = \bar{k}_{\mathrm{p}} = 0$. This is illustrated in Table 1. For more general results non-dimensional frequencies, $\beta = \omega a^2 \left(\frac{\bar{\rho} h}{D}\right)^{0.5}$ are presented in which D and $\bar{\rho}$ are:

$$D = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \frac{E(z)\, z^2}{(1-\upsilon^2)} \mathrm{d}z \text{ and } \bar{\rho} = \frac{1}{h}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \rho(z)\, \mathrm{d}z \qquad (15)$$

From Table 1 it is seen that the results are compatible with those in the literature with great accuracy. A little deviation, however, is seen in higher mode frequencies. The highest value of error can be seen in the 5th mode when θ equals 45, which is less that 2%. This error of the fifth mode is quite acceptable. In fact, the accuracy of the results is proved to decrease for higher skew angles and higher mode shapes [21]. More results for first three normalized frequencies of homogeneous plates with other boundary conditions are presented in Table 2.

Due to lack of valid data for vibration of skew plates on elastic foundation, the results of free vibration of the plate on Winkler-Pasternak foundation have been compared with rectangular results in the literature by setting $\theta = 0$. This is presented in Table 3. It is seen that the present results are remarkably accurate for the fundamental natural frequency.

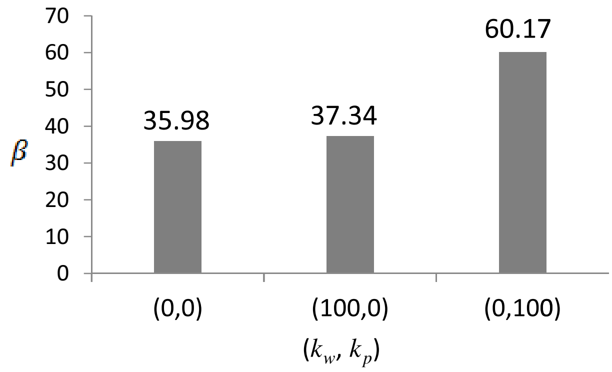

Fig. 3. Effects of Winkler and Pasternak coefficients on frequency parameter for clamped rectangular plate.

3.2 Homogeneous skew plates on Winkler-Pasternak type foundation

In this section, results for normalized natural frequency, β of a homogeneous skew plate on elastic foundation are presented. In Table 4, the first five frequencies of simply supported and clamped skew plates are shown on Winkler-Pasternak elastic foundations. A comparison between Tables 1 and 4 indicates that adding an elastic foundation increases the natural frequency of the plate – as seen in Table 3. More importantly, these frequency changes are more considerable with the changes in Pasternak foundation rather than Winkler foundation. This is clearly shown in Figure 3 when for $\bar{k}_{\mathrm{p}} = 0$ the changes in frequency due to Winkler foundation are not substantial. However, when a Pasternak foundation is set, the changes are more noticeable in natural frequency. It should be mentioned that the selection of foundation coefficients is on the basis of references [23–27], wherein more data for the interaction of structural-foundation of rectangular plates have been presented. More results for the

Table 2. 1st three frequency parameters of the homogeneous skew plate.

θ	method	CCSS			CSSS			SCCC		
		1	2	3	1	2	3	1	2	3
0	Present	27.07	60.61	60.86	23.65	51.70	58.70	31.87	63.50	71.36
	[22]	27.05	60.55	60.55	23.64	51.81	58.65	31.82	63.34	71.08
15	Present	28.69	60.00	68.47	25.05	52.80	64.01	33.81	65.19	77.69
	[20]	28.72	59.74	68.73	25.02	52.76	63.91	33.74	64.96	77.35
30	Present	34.30	66.46	87.36	30.10	58.73	81.14	40.80	73.38	98.37
	[20]	34.54	66.18	87.67	29.94	58.64	80.68	40.61	73.04	97.55
45	Present	48.33	85.39	123.66	42.89	75.22	111.92	58.27	95.72	137.19
	[20]	48.83	85.03	122.97	41.96	74.95	111.29	57.50	94.99	135.35

Table 3. Fundamental frequency of a simply supported homogeneous rectangular plate on elastic foundation.

$(\bar{k}_w,\ \bar{k}_p)$	Akhavan et al. [7]	Atmane et al. [8]	Present
(0,0)	19.7391	19.7391	19.7392
(100,10)	26.2112	26.2112	26.2112
(1000,100)	57.9961	57.9962	57.9961

Table 4. Frequency parameter of a homogeneous skew plate on Winkler-Pasternak foundation.

		SSSS				CCCC			
$(\bar{k}_w,\ \bar{k}_p)$		(0,100)	(0,1000)	(100,0)	(100,100)	(0,100)	(0,1000)	(100,0)	(100,100)
θ	Mode								
0	1	48.616	141.876	22.127	49.634	60.173	151.703	37.3491	60.9978
	2	85.848	227.559	50.351	86.429	104.946	243.129	74.090	105.445
	3	85.848	227.559	50.351	86.429	104.946	243.129	74.090	105.445
	4	118.869	291.875	79.587	119.289	143.421	311.735	108.718	143.843
	5	140.55	329.768	99.806	140.905	168.298	154.012	132.158	168.594
15	1	49.515	143.506	23.086	50.485	61.969	237.825	39.431	62.746
	2	83.546	221.119	49.224	84.142	103.215	257.384	73.574	103.709
	3	92.582	239.519	57.074	93.160	113.997	308.906	83.227	114.450
	4	117.866	287.524	79.832	118.379	143.527	358.188	110.060	143.910
	5	145.078	334.568	105.670	145.813	174.780	327.299	139.701	175.118
30	1	52.802	148.985	26.796	53.615	68.526	161.896	47.027	69.155
	2	85.645	219.928	53.499	86.149	109.293	239.205	82.153	109.688
	3	107.149	259.680	73.134	107.553	134.844	282.184	105.751	135.165
	4	120.184	282.719	85.146	120.544	150.004	306.598	119.835	150.293
	5	164.057	352.049	127.248	164.320	199.014	378.506	167.235	199.232
45	1	61.365	160.914	37.105	61.938	85.411	179.918	66.273	85.824
	2	95.470	226.541	67.041	95.839	129.910	253.406	106.981	130.182
	3	134.408	287.348	104.634	134.671	174.987	318.265	150.563	175.189
	4	141.722	297.744	111.911	141.971	184.209	331.476	159.170	184.401
	5	191.749	363.398	161.726	191.934	232.994	388.393	208.713	233.145

interaction of structure-foundation are available in the Appendix.

When the foundation stiffness is not constant, as it may happen in engineering applications, it is necessary to consider a variable elastic foundation to obtain more reliable results. In this research study, two types of variable foundations including linear and parabolic are discussed. The choice of such foundations is based on Zhou [28] and Pradhan et al. [29] to investigate free vibration of beams on variable elastic foundations. An example of variable foundations can be considered as in Table 5, in which Winkler coefficient is a function of ε and Pasternak coefficient is a function of η.

The changes in frequency parameter for homogeneous skew plates with variable elastic foundations are presented in Figures 4 and 5. All α_is of Table 5 have been set to α in this figure, and it varies from 0 to 1.0 with the intervals of 0.2, and $k_{01} = k_{02} = 1000$. The dependency of the frequency to skew angles, and linearly – column I in Table 5 – and parabolically – column II in Table 5 – variable elastic foundations is illustrated in Figures 4 and 5.

Table 5. Linear and parabolic Winkler-Pasternak foundation coefficients.

	Column I: Linear	Column II: Parabolic
Winkler	$k_w = k_{01}(1 - \alpha_1\varepsilon)$	$k_w = k_{01}(1 - \alpha_2\varepsilon^2)$
Pasternak	$k_p = k_{02}(1 - \alpha_3\eta)$	$k_p = k_{02}(1 - \alpha_4\eta^2)$

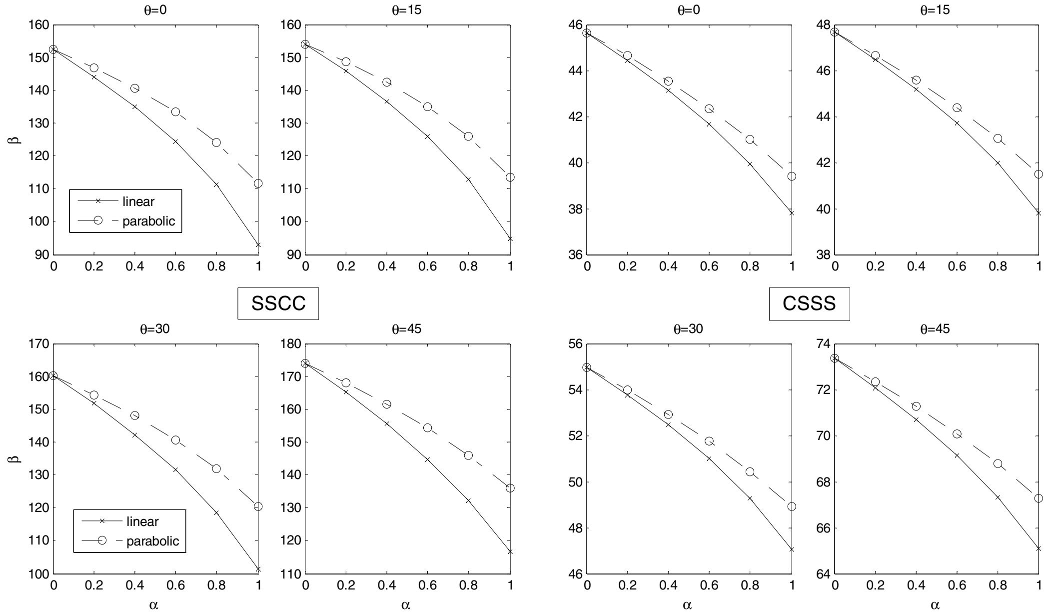

Fig. 4. β vs. variable stiffness coefficients of the foundations of homogeneous skew SSCC (left) and CSSS (right) plates.

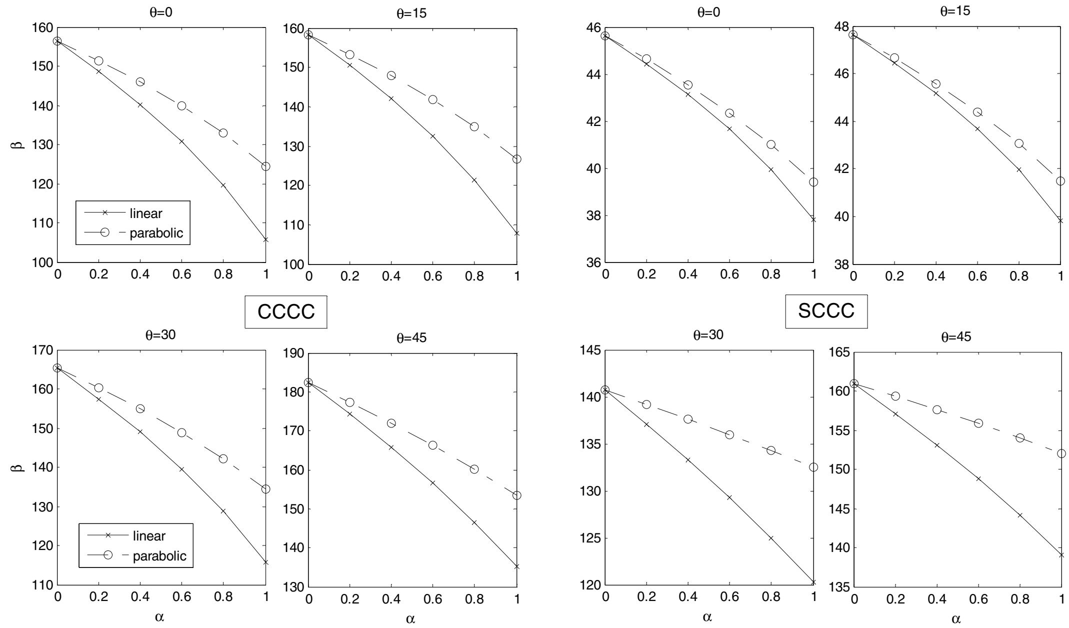

Fig. 5. β vs. variable stiffness coefficients of the foundations of homogeneous skew CCCC (left) and SCCC (right) plates.

Table 6. Material properties of the FGM skew plate.

	Material	E (GPa)	ρ (kg.m^{-3})	υ
Metal	$SUS304$	201.04e9	8166	0.3262
Ceramic	Si_3N_4	348.43e9	2377	0.2400

3.3 FGM skew plates on Winkler-Pasternak foundation

In this section, results for free vibration of FG skew plates on elastic foundation are presented. To do so, material properties are used as in Table 6.

Natural frequencies of FG skew plates on variable Winkler-Pasternak type elastic foundations are presented in Tables 7, 8 and 9 for CCCC, SSSS and CCSS plates, respectively. These tables include two parts. The first set of columns presents the frequencies when only Winkler foundation has been considered, and Pasternak coefficient has been set to zero. The second set of columns presents the frequencies when Winkler and Pasternak foundations, have been considered. The change in variable coefficients is based in Table 5, in which the Winkler foundation changes along ε, while the Pasternak foundation assumes to be constant.

Table 7. Frequency parameter for a skew FGM clamped plate on a linearly variable elastic support.

		$k_p = 0$				$k_p = 100$			
		$k_w = 100(1 - \alpha_1 \varepsilon)$, $\alpha_1 = 0.8$				$k_w = 100(1 - \alpha_1 \varepsilon)$, $\alpha_1 = 0.8$			
θ	n / mode	∞	10	1	0	∞	10	1	0
0	1	36.809	40.583	52.804	87.590	60.669	66.889	87.031	144.366
	2	73.819	81.388	105.896	175.659	105.255	116.047	150.992	250.463
	3	73.820	81.388	105.896	175.660	105.256	116.047	150.992	250.464
	4	108.534	119.661	155.695	258.265	143.705	158.438	206.148	341.956
	5	132.006	145.540	189.366	314.119	168.476	185.749	241.683	400.902
15	1	38.938	42.930	55.858	92.656	62.437	68.839	89.568	148.575
	2	73.311	80.827	105.166	174.448	103.522	114.136	148.505	246.339
	3	82.995	91.503	119.058	197.492	114.281	125.998	163.939	271.941
	4	109.885	121.151	157.633	261.479	143.776	158.516	206.250	342.125
	5	139.563	153.871	200.206	332.100	175.008	192.950	251.053	416.444
30	1	46.657	51.440	66.931	111.024	68.904	75.968	98.845	163.962
	2	81.942	90.343	117.548	194.987	109.530	120.760	157.124	260.636
	3	105.587	116.412	151.467	251.251	135.037	148.881	193.713	321.330
	4	119.690	131.961	171.698	284.811	150.177	165.574	215.433	357.358
	5	167.125	184.259	239.745	397.687	199.144	219.561	285.677	473.878
45	1	66.059	72.832	94.763	157.192	85.659	94.441	122.880	203.833
	2	106.848	117.803	153.277	254.254	130.073	143.408	186.593	309.518
	3	150.469	165.896	215.852	358.053	175.108	193.060	251.196	416.682
	4	159.081	175.390	228.206	378.545	184.324	203.221	264.417	438.613
	5	208.645	230.036	299.307	496.487	233.085	256.981	334.366	554.642

Table 8. Frequency parameter for a skew FGM simply supported plate on a linearly variable elastic support.

		$k_p = 0$				$k_p = 100$			
		$k_w = 100(1 - \alpha_1 \varepsilon)$, $\alpha_1 = 0.8$				$k_w = 100(1 - \alpha_1 \varepsilon)$, $\alpha_1 = 0.8$			
θ	n / mode	∞	10	1	0	∞	10	1	0
0	1	21.202	23.376	30.415	50.452	49.229	54.276	70.621	117.145
	2	49.981	55.105	71.698	118.933	86.219	95.058	123.683	205.164
	3	49.982	55.106	71.700	118.935	86.219	95.058	123.683	205.164
	4	79.378	87.516	113.870	188.887	119.155	131.371	170.931	283.539
	5	99.606	109.818	142.887	237.019	140.763	155.195	201.928	334.957
15	1	22.232	24.511	31.892	52.902	50.101	55.237	71.871	119.218
	2	48.830	53.837	70.048	116.196	83.912	92.515	120.374	199.675
	3	56.735	62.552	81.388	135.005	92.952	102.482	133.342	221.187
	4	79.590	87.749	114.173	189.390	118.216	130.336	169.584	281.303
	5	105.487	116.302	151.323	251.013	145.681	160.616	208.982	346.658
30	1	26.141	28.821	37.500	62.205	53.291	58.755	76.448	126.810
	2	53.174	58.626	76.279	126.531	85.948	94.759	123.294	204.519
	3	72.897	80.370	104.572	173.463	107.392	118.402	154.056	255.546
	4	84.942	93.651	121.852	202.126	120.400	132.744	172.717	286.501
	5	127.112	140.144	182.345	302.472	164.215	181.051	235.570	390.761
45	1	36.722	40.487	52.679	87.383	61.709	68.036	88.524	146.842
	2	66.830	73.682	95.869	159.027	95.692	105.502	137.272	227.706
	3	104.498	115.212	149.906	248.662	134.566	148.362	193.038	320.209
	4	111.785	123.246	160.358	266.001	141.872	156.417	203.518	337.594
	5	161.639	178.210	231.875	384.632	191.860	211.530	275.228	456.545

Table 9. Frequency parameter for a skew FGM CCSS supported plate on a linearly variable elastic support.

		$k_p = 0$				$k_p = 100$			
		$k_w = 100(1 - \alpha_1\varepsilon)$, $\alpha_1 = 0.8$				$k_w = 100(1 - \alpha_1\varepsilon)$, $\alpha_1 = 0.8$			
θ	n	∞	10	1	0	∞	10	1	0
	mode								
0	1	28.072	30.983	40.479	67.410	54.492	60.142	78.576	130.854
	2	61.080	67.412	88.075	146.672	95.215	105.086	137.297	228.642
	3	61.330	67.688	88.436	147.274	95.279	105.157	137.389	228.796
	4	93.329	103.004	134.577	224.112	131.003	144.584	188.902	314.580
	5	115.148	127.086	166.040	276.508	154.134	170.113	222.256	370.126
15	1	29.509	32.568	42.551	70.860	55.734	61.512	80.367	133.835
	2	60.465	66.733	87.188	145.195	93.208	102.871	134.403	223.822
	3	68.873	76.013	99.312	165.386	102.982	113.658	148.496	247.292
	4	94.107	103.863	135.699	225.981	130.523	144.054	188.209	313.427
	5	121.467	134.060	175.151	291.681	159.519	176.057	230.021	383.057
30	1	34.890	38.507	50.310	83.782	60.263	66.510	86.897	144.710
	2	66.829	73.757	96.365	160.478	97.060	107.122	139.957	233.071
	3	87.641	96.727	126.375	210.454	120.104	132.555	173.186	288.408
	4	101.293	111.794	146.061	243.237	134.457	148.396	193.882	322.874
	5	143.918	158.839	207.526	345.594	179.060	197.623	258.198	429.980
45	1	48.532	53.563	69.981	116.541	71.822	79.268	103.565	172.467
	2	85.720	94.606	123.605	205.841	111.750	123.335	161.140	268.348
	3	123.828	136.665	178.556	297.350	151.783	167.518	218.865	364.478
	4	131.311	144.924	189.346	315.320	159.798	176.364	230.423	383.725
	5	168.893	186.402	243.538	405.566	198.063	218.596	285.600	475.612

It was previously mentioned that a pure ceramic plate is achieved for $n = 0$, and a pure metal plate is achieved for $n = \infty$. It is obtained from Tables 7–9 that the frequency increases if one chooses lower values of n, e.g. moving from metal to ceramic. Moreover, as discussed before, the plate frequency is directly dependent to skew angle and Winkler-Pasternak coefficients for different boundary conditions.

4 Conclusion remarks

In this research study, Rayleigh-Ritz formulation was employed to obtain approximate but highly accurate solution for free vibration of skew homogeneous and FGM plates resting on variable elastic foundations. The formulation for the energy of the variable Winkler and Pasternak foundations together with the skew plate was derived. A number of comparisons were made with those available results in the literature and showed an excellent agreement particularly in lower mode frequencies. Finally, the dependency of the free vibration of skew plates on FG properties, skew angles, boundary conditions, foundation stiffness parameters, and linearly or parabolically variable Winkler-Pasternak elastic foundations were investigated. All results presented in this paper may serve to validate other analytical and numerical methods. In future research, it would be beneficial to study the free vibration analysis of piezoelectric skew plates resting on elastic foundations

Appendix

More results for the interaction of structure-foundation of homogeneous plates are presented in the following tables.

Table A.1. Frequency parameter of a homogeneous skew CCSS plate on Winkler-Pasternak foundation.

$(\bar{k}_w, \bar{k}_p)$		(0,100)	(0,1000)	(100,0)	(100,100)	(10,10)	(100,1000)
θ	Mode						
0	1	53.968	147.290	28.858	54.887	31.078	147.629
	2	94.913	236.326	61.433	95.439	64.979	236.537
	3	94.975	236.328	61.673	95.500	65.178	236.540
	4	130.781	303.203	93.554	131.163	97.527	303.368
	5	153.945	342.299	115.330	154.270	119.428	342.445
15	1	55.240	149.200	30.234	56.108	32.494	149.524
	2	92.909	229.956	60.802	93.427	64.157	230.166
	3	102.712	249.087	69.176	103.181	72.727	249.281
	4	130.307	299.042	94.321	130.677	98.136	299.204
	5	159.343	347.267	121.634	159.646	125.616	347.406
0	1	59.856	155.707	35.445	60.576	37.749	155.985
	2	96.803	229.941	67.106	97.250	70.166	230.129
	3	119.898	271.351	87.855	120.258	91.210	271.510
	4	134.270	295.305	101.472	134.592	104.895	295.452
	5	178.922	364.796	144.046	179.164	147.675	364.915
45	1	71.547	170.280	48.861	72.040	51.078	170.488
	2	111.570	240.002	85.898	111.886	88.492	240.149
	3	151.648	301.943	123.949	151.881	126.775	302.060
	4	159.672	313.780	131.428	159.893	134.323	313.893
	5	197.959	366.779	168.980	198.138	171.937	366.875

Table A.2. Frequency parameter of a homogeneous skew CSSS plate on Winkler-Pasternak foundation.

$(\bar{k}_w, \bar{k}_p)$		(0,100)	(0,1000)	(100,0)	(100,100)	(10,10)	(100,1000)
θ	Mode						
0	1	30.215	42.414	24.336	30.289	25.188	42.422
	2	56.850	66.442	52.004	56.889	52.769	66.447
	3	81.945	97.915	66.403	82.042	70.858	97.925
	4	105.529	113.077	91.895	105.549	99.801	113.080
	5	106.923	120.545	101.250	106.949	101.935	120.552
15	1	31.989	44.478	25.828	32.057	26.968	44.487
	2	59.725	69.093	54.160	59.755	55.658	69.097
	3	87.113	103.950	70.353	87.230	75.036	103.959
	4	103.766	112.605	90.990	103.794	97.619	112.608
	5	118.876	133.925	109.874	118.933	111.870	133.932
0	1	38.545	51.952	31.119	38.597	33.297	51.959
	2	70.203	79.109	61.206	70.223	64.957	79.112
	3	105.398	123.438	85.454	105.563	90.782	123.443
	4	111.870	125.503	95.665	111.931	102.914	125.514
	5	146.137	169.220	133.265	146.242	136.315	169.231
45	1	55.466	70.622	44.202	55.498	48.508	70.629
	2	95.530	105.757	77.582	95.570	84.612	105.759
	3	135.673	158.850	114.616	135.785	121.552	158.861
	4	150.686	178.146	125.599	150.860	133.383	178.161
	5	184.465	240.065	158.207	184.586	164.008	240.088

Table A.3. Frequency parameter of a homogeneous skew SCCC plate on Winkler-Pasternak foundation.

$(\bar{k}_w,\ \bar{k}_p)$		(0,100)	(0,1000)	(100,0)	(100,100)	(10,10)	(100,1000)
θ	Mode						
0	1	52.509	125.538	33.306	52.996	35.442	125.728
	2	92.170	185.294	66.409	92.405	72.366	185.401
	3	111.183	259.795	72.006	111.632	74.469	259.866
	4	146.976	280.951	103.182	147.271	109.343	281.117
	5	151.615	337.912	122.101	151.751	133.850	338.037
15	1	54.628	127.947	35.174	55.081	37.570	128.128
	2	95.850	188.793	67.601	96.073	73.074	188.895
	3	115.264	265.663	78.700	115.687	82.107	265.732
	4	146.439	285.319	104.236	146.699	110.848	285.478
	5	163.208	340.253	130.548	163.367	141.173	340.376
0	1	62.288	136.461	41.931	62.645	45.171	136.614
	2	107.917	201.527	75.270	108.124	81.396	201.617
	3	131.386	285.662	99.393	131.720	103.656	285.729
	4	154.029	300.577	112.638	154.275	120.039	300.715
	5	194.433	351.254	156.070	194.559	166.001	351.365
45	1	81.375	157.180	59.017	81.600	63.743	157.290
	2	133.035	233.422	96.981	133.220	104.167	233.492
	3	176.266	323.111	138.507	176.511	146.281	323.191
	4	179.832	337.494	148.603	179.984	155.221	337.593
	5	232.753	394.920	186.267	232.900	197.241	394.988

References

[1] M. Koizumi, FGM activities in Japan, Compos. Part B Eng. 8368 (1997) 1–4

[2] R.K. Kaul, V. Cadambe, The natural frequencies of thin skew plates, Aeronaut. Q. 7 (1956) 337–352

[3] C.M. Wang, K.K. Ang, L. Yang, E. Watanabe, Free Vibration of Skew Sandwich Plates With Laminated Facings, J. Sound Vib. 235 (2000) 317–340

[4] K.S. Woo, C.H. Hong, P.K. Basu, C.G. Seo, Free vibration of skew Mindlin plates by p-version of FEM, J. Sound Vib. 268 (2003) 637–656

[5] X. Zhao, Y.Y. Lee, K.M.Ã. Liew, Free vibration analysis of functionally graded plates using the element-free kp -Ritz method, J. Sound Vib. 319 (2009) 918–939

[6] Ö. Civalek, Nonlinear analysis of thin rectangular plates on Winkler-Pasternak elastic foundations by DSC-HDQ methods, Appl. Math. Model. 31 (2007) 606–624

[7] H. Akhavan, S. Hosseini Hashemi, H.R. Damavandi Taher, a. Alibeigloo, S. Vahabi, Exact solutions for rectangular Mindlin plates under in-plane loads resting on Pasternak elastic foundation. Part I: Buckling analysis. Comput. Mater. Sci. 44 (2009) 968–978

[8] H.A. Atmane, A. Tounsi, I. Mechab, E.A.A. Bedia, Free vibration analysis of functionally graded plates resting on Winkler-Pasternak elastic foundations using a new shear deformation theory, Int. J. Mech. Mater. Des. 6 (2010) 113–121

[9] A. Joodaky, I. Joodaky, M. Hedayati, R. Masoomi, E. Borzabadi, Deflection and stress analysis of thin FGM skew plates on Winkler foundation with various boundary conditions using extended Kantorovich method, Compos. Part B. 51 (2013) 191–196

[10] A. Fekrar, M.S.A. Houari, A. Tounsi, S.R. Mahmoud, A new five-unknown refined theory based on neutral surface position for bending analysis of exponential graded plates, Meccanica 49 (2014) 795–810

[11] Y. Benveniste, A new approach to the application of Mori-Tanaka's theory in composite materials, Mech. Mater. 6 (1987) 147–157

[12] Z. Belabed, M.S. Ahmed Houari, A. Tounsi, S.R. Mahmoud, O. Anwar Bég, An efficient and simple higher order shear and normal deformation theory for functionally graded material (FGM) plates, Compos. Part B Eng. 60 (2014) 274–283

[13] S.A. Yahia, H.A. Atmane, M.S.A. Houari, A. Tounsi, Wave propagation in functionally graded plates with porosities using various higher-order shear deformation plate theories, Struct. Eng. Mech. 53 (2015) 1143–1165

[14] A.A. Bousahla, M.S.A. Houari, A. Tounsi, E.A. Adda Bedia, a Novel Higher Order Shear and Normal Deformation Theory Based on Neutral Surface Position for Bending Analysis of Advanced Composite Plates, Int. J. Comput. Methods. 11 (2014) 1350082

[15] L.O. Larbi, A. Kaci, M.S.A. Houari, A. Tounsi, An Efficient Shear Deformation Beam Theory Based on Neutral Surface Position for Bending and Free Vibration of Functionally Graded Beams #, Mech. Based Des. Struct. Mach. 41 (2013) 421–433

[16] S.A . Eftekhari, A.A. Jafari, Modified mixed Ritz-DQ formulation for free vibration of thick rectangular and skew plates with general boundary conditions, Appl. Math. Model. 37 (2013) 7398–7426

[17] R.B. Bhat, Natural frequencies of rectangular plates using characteristic orthogonal polynomials in Rayleigh-Ritz method, J. Sound Vib. 102 (1985) 493–499

[18] O. Beslin, J. Nicolas, A hierarchical functions set for predicting very high order plate bending modes with any boundary conditions, J. Sound Vib. 202 (1997) 633–655

[19] J. Yang, H. Shen, Dynamic response of initially stressed functionally graded rectangular thin plates, Compos. Struct. 54 (2001) 497–508

[20] X. Wang, Y. Wang, Z. Yuan, Accurate vibration analysis of skew plates by the new version of the differential quadrature method, Appl. Math. Model. 38 (2014) 926–937

[21] A.W. Leissa, Vibration of Plates, NASA SP-160 (1969)

[22] B. Singh, S. Chakraverty, Flexural vibration of skew plates using boundary characteristic orthogonal polynomials in two variables, J. Sound Vib. 173 (1994) 157–178

[23] M.A.A. Meziane, H.H. Abdelaziz, A. Tounsi, An efficient and simple refined theory for buckling and free vibration of exponentially graded sandwich plates under various boundary conditions, J. Sandw. Struct. Mater. 16 (2014) 293–318

[24] K. Nedri, N. El Meiche, A. Tounsi, Free vibration analysis of laminated composite plates resting on elastic foundations by using a refined hyperbolic shear deformation theory, Mech. Compos. Mater. 49 (2014) 629–640

[25] B. Bouderba, M.S.A. Houari, A. Tounsi, Thermomechanical bending response of FGM thick plates resting on Winkler-Pasternak elastic foundations, Steel Compos. Struct. 14 (2013) 85–104

[26] S. Motaghian, M. Mofid, J.E. Akin, On the free vibration response of rectangular plates , partially supported on elastic foundation, Appl. Math. Model. 36 (2012) 4473–4482

[27] S.A. Sheikholeslami, A.R. Saidi, Vibration analysis of functionally graded rectangular plates resting on elastic foundation using higher-order shear and normal deformable plate theory, Compos. Struct. 106 (2013) 350–361

[28] D. Zhou, A general solution to vibrations of beams on variable Winkler elastic foundation, Comput. Struct. 47 (1993) 83–90

[29] S.C.Ã. Pradhan, T. Murmu, Thermo-mechanical vibration of FGM sandwich beam under variable elastic foundations using differential quadrature method, J. Sound Vib. 321 (2009) 342–362