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**Experimental Vibration Analysis**

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# Order tracking using $H_\infty$ estimator and polynomial approximation

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**Abstract.** This paper presents the $H_\infty$ estimator for discrete-time varying linear system combined with the polynomial approximation for order tracking of non-stationary signals. The proposed approach is applied to the gearbox diagnosis under variable speed condition. In this instance, it is well known that the occurrence of a fault on a gear tooth leads to an amplitude and a phase modulation in the vibration signal. The purpose is to estimate this unknown amplitude and phase modulation by tracking orders. To estimate these modulations, the vibration signal is described in state space model. Then, the $H_\infty$ criterion is used to minimize the worst possible amplification of the estimation error related to both the process and the measurement noises. Such an approach doesn't require any assumption on the statistic properties of the noises unlike the Kalman estimator. A numerical example is given in order to evaluate the performance of the $H_\infty$ estimator regarding the conventional Kalman estimator.

**Keywords:** Order tracking / estimator / polynomial approximation / non-stationary conditions

## 1 Introduction

Order tracking using the state space approach is one of the tools widespread for the processing of non-stationary signals. The so-called the state space model is composed of two equations: the state equation and the measurement equation. The technique the most presented in this area is the Kalman estimator and more precisely the Vold-Kalman estimator in the area of the mechanical systems diagnosis [1]. Vold et al. present the theoretical basis about this estimator in [2]. This kind of estimator supposes that the measurement noise and the process noise are centered, Gaussian and white with known statistics.

In the literature many works on the Vold-Kalman estimator for order tracking have been done. Pan and Lin have realized an interesting explorative study on the Vold-Kalman estimator [3]. Behrouz and al. also applied this estimator to diagnose a bearing default and they have translated the state equation in term of second order autoregressive model [4]. These study have provided conclusive results. However, the unrealistic assumptions on the noises naturally limit the application of this estimator in real cases.

Therefore, the $H_\infty$ estimator is proposed in this paper to evaluate the amplitude modulation and the phase modulation. To estimate these modulations the vibration signal is described using the state variables. Then these latters are modelled by a Taylor series. This method generalizes that of Vold-Kalman. With the $H_\infty$ estimator we make no assumption on the noise statistics. They must only be of finite energy. More details on the discrete $H_\infty$ estimator can be found in the work of Shen and Deng [5].

This paper is structured as follows: Section 2 presents the theoretical foundation about the $H_\infty$ estimator and Section 3 provides an example of simulation which validated our proposal.

## 2 Theoretical background

### 2.1 Problem formulation

In this paper, the gearbox vibration signal is modelled as

$$y(t) = \sum_{i=1}^{M} A_i(t)\cos\left(2\pi\int_0^t f_i(u)du + \varphi_i(t)\right) + v(t) \qquad (1)$$

where $A_i$ and $\varphi_i$ are respectively the amplitude and the phase of the $i$th order, $v$ is the measurement noise which contains the unwanted part of the signal, $f_i = o_i f_r$ is the instantaneous frequency of the order of interest with $f_r$ the reference frequency and $o_i$ the value of the order $i$.

In the discrete form, (Eq. (1)) becomes:

$$y(k) = \sum_{i=1}^{M} A_i(k)\cos(\theta_i(k) + \phi_i(k)) + v(k), \quad k = 0, 2, \ldots, n-1, \qquad (2)$$

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where $\theta_i(k) = 2\pi \sum_{j=1}^{k} \frac{f_i(j)}{f_s}$ is the angular displacement and $f_s$ is the sampling frequency.

The purpose is to estimate the amplitude and the phase of some specific orders of interest using the $H_\infty$ estimation approach. For this, the problem is formulated in term of estimation of the state variables. Note that the amplitude and the phase modulation are the key features to diagnose the gear state [6].

## 2.2 State space modelling

Let us consider the formula established in (Eq. (2)). The purpose here is to build the measurement and the state equation.

Linearizing (Eq. (2)) leads to:

$$y(k) = \sum_{i=1}^{M} \left[ \cos(\theta_i(k)) \quad -\sin(\theta_i(k)) \right] \begin{bmatrix} a_{i,c}(k) \\ a_{i,s}(k) \end{bmatrix} + v(k), \quad (3)$$

where $a_{i,c} = A_i \cos\varphi_i$ and $a_{i,s} = A_i \sin\varphi_i$. Let put $a_i(k) = \begin{bmatrix} a_{i,c}(k) \\ a_{i,s}(k) \end{bmatrix}$ and $B_i(k) = \left[ \cos(\theta_i(k)) \quad -\sin(\theta_i(k)) \right]$.

The amplitudes $a_{i,c}$ and $a_{i,s}$ are unknown. For estimating them, these amplitudes are modeled by a polynomial approximation as follows:

$$a_{i,c}(k) = \sum_{q=0}^{N} \alpha_{i,c}^{q}(k) t^q(k), \quad (4)$$

$$a_{i,s}(k) = \sum_{q=0}^{N} \alpha_{i,s}^{q}(k) t^q(k), \quad \mathrm{i} = 1, 2, \ldots, M \quad (5)$$

and the coefficients of the polynomial by a random walk process such as:

$$\alpha_{i,c}^{q}(k+1) = \alpha_{i,c}^{q}(k) + w_{i,c}(k), \quad (6)$$

$$\alpha_{i,s}^{q}(k+1) = \alpha_{i,s}^{q}(k) + w_{i,s}(k), \quad (7)$$

where $w_{i,.}$ is a random signal. With those new variables (3) can be rewritten as:

$$y(k) = \sum_{i=1}^{M} B_i(k) \tilde{T}(k) x_i(k) + v(k), \quad (8)$$

with $\tilde{T}(k) = \left[ T(k) \quad T(k) \right]^T$, $T(k) = \left[ 1 \quad t(k) \quad \cdots \; t^N(k) \right]$ and $x_i(k) = \begin{bmatrix} x_{i,c} \\ x_{i,s} \end{bmatrix}$ with $x_{i,.} = \left[ \alpha_{i,.}^0 \quad \alpha_{i,.}^1 \quad \cdots \quad \alpha_{i,.}^N \right]^T$. Note that $A^T$ is the transpose of the matrix $A$.

Assuming that the measurement matrix is $H(k) = \left[ B_1(k) T(k) \quad B_2(k) T(k) \quad \cdots \quad B_M(k) T(k) \right]$, the following measurement equation is obtained:

$$y(k) = H(k) x(k) + v(k), \quad (9)$$

where $x(k) = \left[ x_1(k) \quad x_2(k) \quad \cdots \quad x_M(k) \right]^T$ and $v$ is the measurement noise with a covariance matrix $V$.

Then the state equation is:

$$x(k+1) = F x(k) + w(k), \quad (10)$$

where $F = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$ and $w(k) = \left[ w_{1,c} \quad w_{1,s} \quad w_{2,c} \quad w_{2,s} \quad \cdots \quad \cdots \quad w_{M,c} \quad w_{M,s} \right]^T$ is the process noise with a covariance matrix $W$.

## 2.3 Discrete $H_\infty$ estimator design

From equations (9) and (10) the following state space model is considered:

$$\begin{cases} x_{k+1} = & F x_k + B w_k \\ y_{k+1} = & H_k x_k + v_k \end{cases}. \quad (11)$$

Let us note $e_k = x_k - \hat{x}_k$ the estimation error where $\hat{x}_k$ is the estimate of $x_k$ and $E\{\,.\,\}$ will stand for the expectation value.

Several facts may be used against the Kalman estimator although it is an attractive and powerful tool to estimate $x_k$;

- the Kalman estimator minimizes $E\left\{ e_k e_k^T \right\}$ while the user may be interested in minimizing the worst-case error;
- the Kalman estimator assumes that the noises are zero-mean with Gaussian distribution;
- the Kalman estimator assumes also that $E\left\{ v_k v_k^T \right\}$ and $E\left\{ w_k w_k^T \right\}$ are known.

These limitations have led to the statement of the $H_\infty$ estimation problem. Several formulations exist in the literature. The $H_\infty$ estimator solution that we present here is originally developed by Ravi Banar [7] and further explored by Shen and Deng [5]. These pioneers define the following cost function:

$$J = \frac{\sum_{k=0}^{n-1} \| x_k - \hat{x}_k \|_Q^2}{\| x_0 - \hat{x}_0 \|_{P_0^{-1}}^2 + \sum_{k=0}^{n-1} \left( \| w_k \|_{W^{-1}}^2 + \| v_k \|_{V^{-1}}^2 \right)} \quad (12)$$

where $\hat{x}_0$ is an estimate of $x_0$, $Q > 0$, $P_0 > 0$, $W > 0$ and $V > 0$ are the weighting matrices and are left to the choice of the designers and depend on the performance requirements. The notation $\| x_k \|_Q^2$ defines the weighted $Q - L_2$ norm, i.e, $\| x_k \|_Q^2 = x_k^T Q x_k$.

Problem statement [8]: Given the scalar $\gamma > 0$, find estimation strategy that achieve

$$sup\, J < 1/\gamma \quad (13)$$

where "*sup*" is the supremum value and $\gamma$ is the desired level of noise attenuation.

The $H_\infty$ estimation problem consists of the minimization of the worst possible amplification of the estimation error. This can be interpreted as a "minmax" problem in which the estimation error is to be minimized and the exogenous disturbances ($w_k$ and $v_k$) and the error of initialization ($x_0 - \hat{x}_0$) are to be maximized.

Remember that unlike the Wiener/Kalman estimator, the $H_\infty$ estimator deals with deterministic noises and no *a*

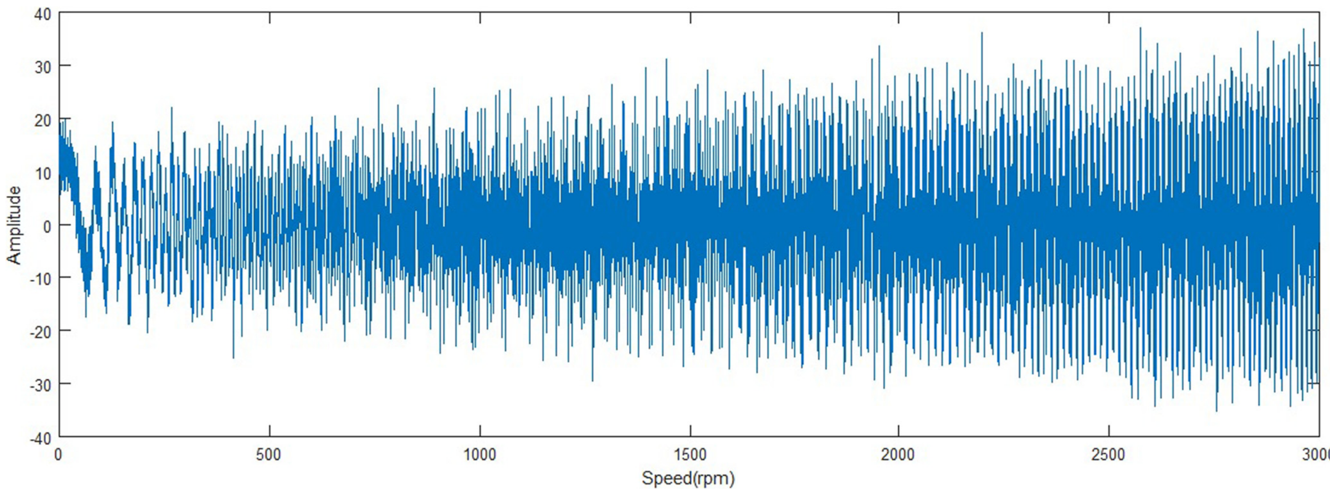


**Fig. 1.** Synthetic signal.

**Table 1.** The amplitudes of the synthetic signal.

| Order number | 1 | 4 | 9 |
|---|---|---|---|
| Amplitude | Linearly increasing from 0 to 10 | Linearly increasing from 3 to 13 | Fixed at 10 |

*priori* information on their statistic properties are required. The solution of the $H_\infty$ estimation problem is given in the theorem below from [5].

Theorem: Let $\gamma > 0$ be a prescribed level of noise attenuation. Then, there exists a $H_\infty$ estimator for $x_k$ if and only if there exists a stabilizing symmetric solution $P_k > 0$ to the following discrete-time Riccati equation:

$$P_{k+1} = FP_k\left(I - \gamma QP_k + H_k^T V^{-1} H_k P_k\right)^{-1} F^T + BWB^T. \tag{14}$$

Then the $H_\infty$ estimator gives the estimate $\hat{x}_k$ of $x_k$ such as:

$$\hat{x}_{k+1} = F\hat{x}_k + K_k(y_k - H_k\hat{x}_k), \hat{x}_0 = x_0. \tag{15}$$

$K_k$ is the gain of the $H_\infty$ estimator and is given by:

$$K_k = FP_k\left(I - \gamma QP_k + H_k^T V^{-1} H_k P_k\right)^{-1} H_k^T V^{-1}. \tag{16}$$

Another way to solve the Riccati equation (14) is presented by Yaesh and Shaked [9]. The method is given as follows:

1. Form the Hamiltonian

$$Z = \begin{bmatrix} F^{-T} & F^{-T}[H^T R^{-1} H - \gamma I] \\ BQB^T F^{-T} & F + BQB^T F^{-T}[H^T R^{-1} H - \gamma I] \end{bmatrix} \in \mathbb{R}^{2n*2n}, \tag{17}$$

where $n$ is $x$ dimension.

2. Find the eigenvectors of eigenvalues $\mathscr{E}_i$ $(i = 1, \ldots, n)$ corresponding to the outside the unit circle

3. Form the matrix of the corresponding eigenvectors denoted by:

$$(\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n) \equiv \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix}, \quad \mathcal{X}_1 \mathcal{X}_2 \in \mathbb{R}^{n*n}. \tag{18}$$

4. Compute $P = \mathcal{X}_2 \mathcal{X}_1^{-1}$.

Note that the smaller $\gamma$, the more easy the problem is to solve. When $\gamma$ tends to $\gamma_{\text{opt}}$ (the optimal value of $\gamma$) the eigenvalues of $P$ tend to infinity and therefore $\mathcal{X}_1$ is close to a singular matrix. Shaked and Theodor [10] investigated the behavior of the optimal $H_\infty$ estimator when $\gamma$ tends to $\gamma_{\text{opt}}$. They showed that when $\gamma$ reaches $\gamma_{\text{opt}}$, there exists at least one or more unbounded eigenvalues.

In the special case, where $\gamma \to 0$, the $H_\infty$ estimator reduces to a Kalman estimator.

## 3 Numerical implementation

In this section, a synthetic signal is used to illustrate the performances of $H_\infty$ estimation approach. The generated signal (see Fig. 1) is described by the following equation.

$$y(t) = \sum_{i=1}^{3} A_i(t) \cos\left(2\pi o_i \int_0^t f_r(u)du\right) + v(t), \tag{19}$$

where $f_r$ is the instantaneous frequency linearly increasing from 0 to 50 Hz in 5 s, $o_i$ contains the order's number and $v$ is the measurement noise. The signal is composed of three orders presented in the Table 1. Figure 2 displays the rpm-frequency spectrum using the conventional windowing Fourier transform that characterizes three orders.

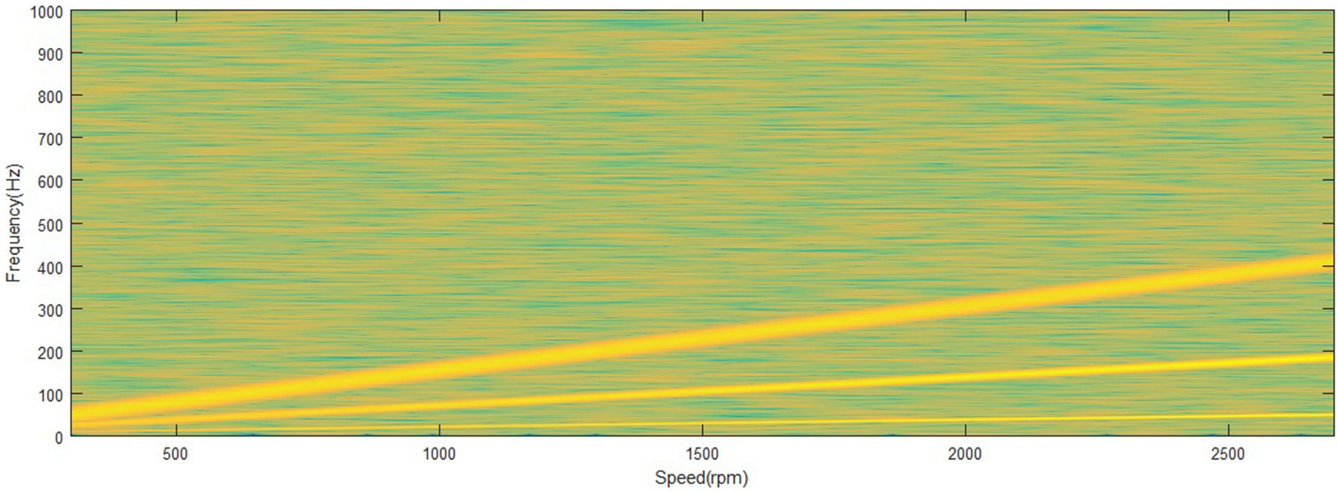


**Fig. 2.** Illustration of rpm-frequency spectrum.

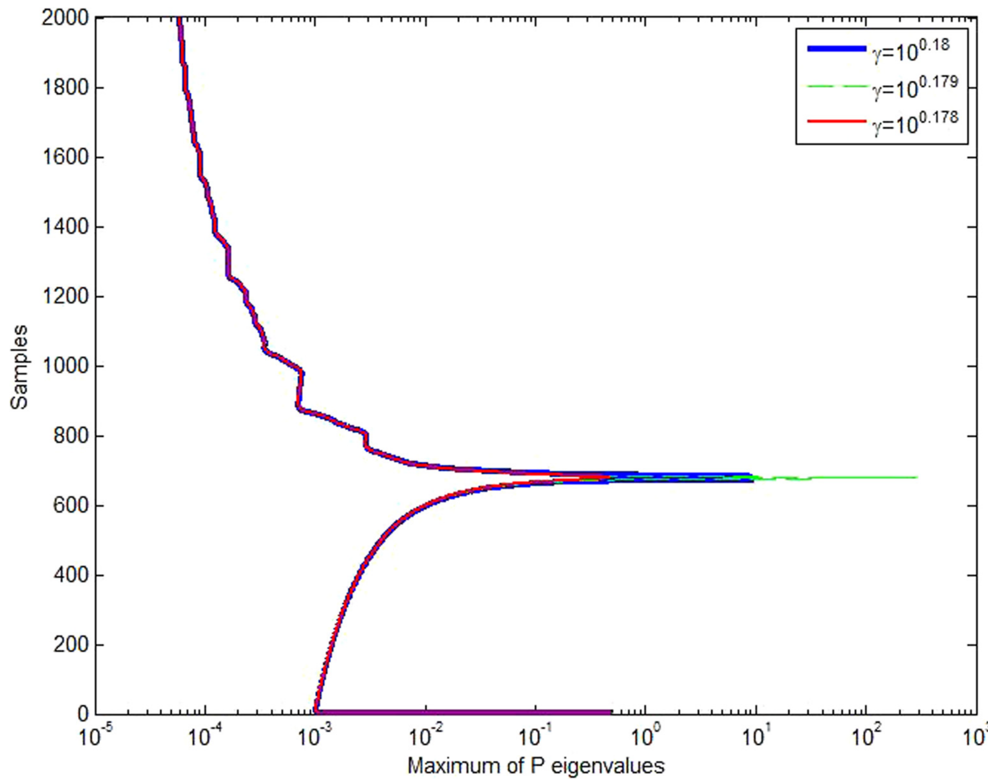


**Fig. 3.** Maximum of the eigenvalues of the covariance matrix error.

The results presented below have been got using a Monte-Carlo simulation based on 400 iterations.

The parameters of the estimator have been taken as follows:

– the covariance of the process noise $W = 10^{-9}$;
– the covariance of the measurement noise $V = 10^{-3}$;
– the initial covariance error $P_0 = 10^{-3}$;
– the level of the noise attenuation $\gamma = \gamma_{\text{opt}} = 10^{0.178}$.

$\gamma_{\text{opt}}$ is equal to the greatest value that guarantees the stability of the matrix $P$. This stability is reached, according to Yaesh and Shaked [11], when $P$s eigenvalues are bounded in the unit circle. As plotted in the Figure 3, this stability is reached for $\gamma = 10^{0.178}$. Beyond this value there exists at least one or more eigenvalues that are outside the unit circle.

The measurement noise is modelled by a Poisson noise as mentioned in [11]. The Kalman estimator algorithm presented by Dan Simon [12] and the $H_\infty$ estimator have been applied to the generated vibration signal. The performance of both estimators is measured in term of signal to noise ratio. Table 2 gives the performance got for the two estimators. In both cases the $H_\infty$ estimator provides a better result than the

**Table 2.** Performance comparison between Kalman and $H_\infty$ filtering.

| | Estimation algorithm | $SNR_{\text{out}}$ | |
|---|---|---|---|
| | | White Gaussian noise | Poisson noise |
| 5 dB | Kalman | 29.9771 | 23.3424 |
| | $H_\infty$ | 30.7015 | 23.7324 |
| 10 dB | Kalman | 39.9468 | 33.2658 |
| | $H_\infty$ | 40.6510 | 33.8200 |
| 15 dB | Kalman | 49.2331 | 43.0675 |
| | $H_\infty$ | 49.9383 | 44.0972 |

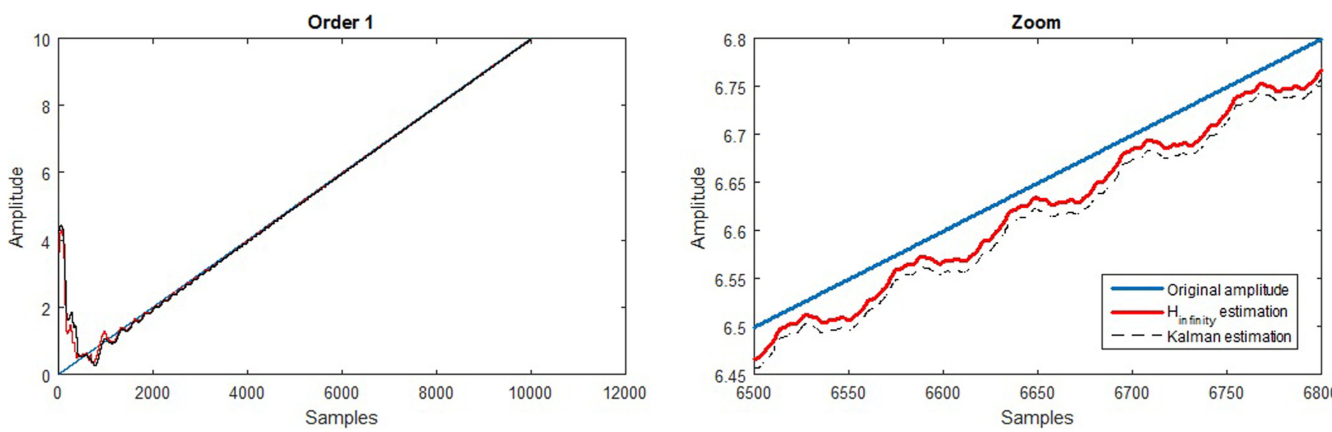


**Fig. 4.** Amplitude of the 1st order estimated using the $H_\infty$ and the Kalman estimator.

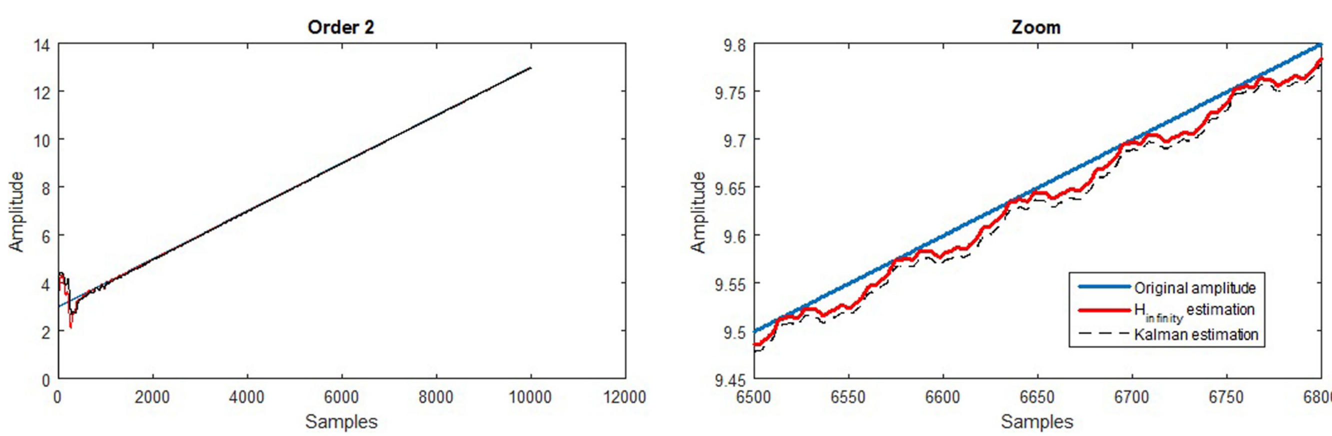


**Fig. 5.** Amplitude of 3rd order estimated using the $H_\infty$ and the Kalman estimator.

Kalman estimator. The $SNR_{\text{out}}$ value is the signal to noise ratio calculated by

$$SNR_{\text{out}} = 10 * log_{10} \frac{\sum_{k=1}^{N} y_k^2}{\sum_{k=1}^{N} (y_k - \hat{y}_k)^2} \quad (20)$$

where $N$ is the number of samples, $y_k$ is the noiseless signal at times $k$ and $\hat{y}_k$ is the estimated or filtered signal. The criterion of comparison is improved by about 0.7 dB using the $H_\infty$ estimator. Therefore the $H_\infty$ estimator is a good alternative to deal with real situation where the noises are not really Gaussian.

Figures 4–6 show the effectiveness of the $H_\infty$ estimator for order tracking in non-stationary signal processing. We see in this last figure that the estimated we got by the $H_\infty$ estimation is closer to the original amplitude than the Kalman estimation.

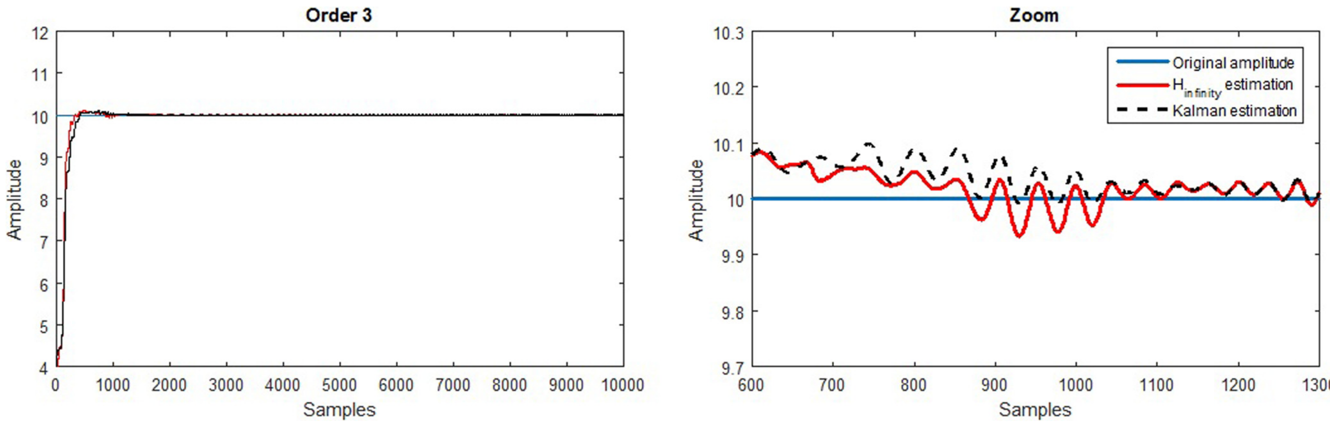


**Fig. 6.** Amplitude of the 9th order estimated using the $H_\infty$ and the Kalman estimator.

## 4 Conclusion

Through this paper a method has been developed to estimate order's amplitude based on the $H_\infty$ estimation in non-stationary operations. This method uses the information of the instantaneous frequency of the signal and makes no assumption on the noises statistics. It takes advantage on the classical Kalman estimation and it can be considered as an extension of this last one. Since the estimator is designed to minimize the worst case-disturbances, the $H_\infty$ estimation approach is more robust to process any kind of noisy signal. The application of this method in real-life data will concern our future research.

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