Effects of molecule force on free vibration for a micro electromagnetic harmonic drive system

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Abstract. In this paper, a micro electromagnetic harmonic drive system is proposed. Considering Van der Waals force, dynamics equation of the flexible ring for the micro drive system is deduced and resolved. Using the equations, the effects of the molecule force on the natural frequencies and vibration modes of the drive system are investigated. Results show that considering molecule force, natural frequencies of the flexible ring are reduced and its vibration modes are changed. For lower order modes, smaller clearance between the flexible ring and stator, smaller thickness of the flexible ring and larger radius of the flexible ring, the effects of the molecule force on the natural frequencies and vibration modes are more obvious.

Keywords: Electromagnetic harmonic drive system / micro drive / Van der Waals force / free vibration / natural frequency / vibration modes

1 Introduction

Electromagnetic harmonic drive is a kind of mechatronics drive device combining rotating magnetic field with harmonic drive. It is widely used in the driving mechanism of machine tools and instruments, and is suitable for submarine navigation, aerospace and transportation fields [1–5].

Herdeg proposed the electromagnetic harmonic drive, and successfully developed the experimental prototype of the drive with external magnetic poles [6]. Janes developed an electromagnetic harmonic drive system with built-in electromagnetic winding, which is compact in structure and can transmit more torque per unit space [7–9]. Rens developed a new type of permanent magnet harmonic drive prototype, which is suitable for the requirements of large transmission ratio [10,11]. Tjahjowidodo proposed a harmonic drive model using statistical measures of variation and analyzed its reliability under different conditions [12,13]. Reinhard investigated small harmonic gear drive and developed the harmonic reducer with metal gears which was used in robot driver for semiconductor chip packaging [14,15]. Jose designed harmonic drive with low temperature magnetic superconductor to improve the service life [16]. Chigira proposed a harmonic drive with a stackable structure that is easy to assemble. By adjusting the structure of the magnetic gear, the maximum transfer

torque can be increased [17]. Based on studies of the electromagnetic harmonic drive, Xu proposed an electromagnetic harmonic movable tooth drive system and investigated its output torque [18–20]. Afanas'ev calculated the electromagnetic moments of an electromagnetic gear reducer by the energy method [21]. Ando proposed a new harmonic gear with stackable structure easy to assemble and studied the effect of stack on the maximum transmit torque by experiments [22]. Koji proposed a new magnetic harmonic gear which has the stackable structure in which the maximum transmission torque of the gear was improved approximately 5.96 times and the torque density was improved about 3.82 times [23]. Liu et al. investigated eccentric harmonic magnetic gear and presented an analytical method for predicting the distribution of magnetic field in the air gap of harmonic gear [24]. Jing proposed a new type of eccentric harmonic magnetic gear and calculated corresponding magnetic field and static torque with the finite element analysis [25]. With the development of MEMS technology, the size of the driving link is more and more limited, and the micron scale drive technology is urgently needed [26].

Therefore, the Authors propose a micro electromagnetic harmonic drive system. It has advantages such as small volume, light weight, simple and compact structure, large speed ratio and small inertia, etc. Stator components with electrode segments and special piezoelectric ceramic materials are not needed, which is more conducive to the miniaturization.

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Fig. 1. An electromagnetic micro harmonic drive system. 1 Cover; 2 flexible ring; 3 stator. (a) Section diagram (b) 3D model.

Fig. 2. Electromagnetic and molecular forces on the flexible ring. (a) Flexible ring and stator (b) electromagnetic force (c) molecular force.

In the drive system, the electromagnetic force has a decisive role for its operation behavior. However, with further reducing the size of the system, the effect of molecular force is becoming more and more significant. The Van der Waals force is an important molecular force, will significantly influence the dynamic performance of the micro electromagnetic harmonic drive system.

In this paper, considering Van der Waals force, dynamics equation of the flexible ring for the micro drive system is deduced and resolved. Using the equations, effects of the molecule force on the natural frequencies and vibration modes of the drive system are investigated. Results show that considering molecule force, natural frequencies of the flexible ring are reduced and its vibration modes are changed. The research is useful in design of the dynamics performance for the micro electromagnetic harmonic drive system.

2 Structure and operation principle

Figure 1 shows a micro electromagnetic harmonic drive system. It consists of micro flexible ring and stator. When the external magnetic field is applied sequentially, the rotating magnetic field will cause the flexible wheel to undergo periodic elastic deformation, thus driving the associated supporting shaft to rotate.

Here, r is radius of the flexible ring, l is the length of the flexible ring, t_0 is the clearance between the flexible ring and stator. The stator material is no magnetic and the flexible ring material is magnetic. The flexible ring under electromagnetic force and molecule force is given in Figure 2. In electromagnetic field, distributed electromagnetic force occurs on the flexible ring. q_{re} is electromagnetic force per unit length. Meanwhile, molecule force occurs at the angle range $[-\frac{\pi}{4}, \frac{\pi}{4}]$. q_{r3} is Van der Waals force per unit length.

3 Static displacement of the flexible ring

The electromagnetic force per unit length on the flexible wheel is

$$q_{re} = \frac{n_B^2 \mu_0 R_I^4 l}{8(R_I^2 + x_B^2)^3} I^2 \cos \theta \quad (1)$$

where μ_0 –vacuum permeability, $\mu_0 = 4\pi \times 10^{-7} \text{N} \cdot \text{A}^{-2}$;
 I –current in magnetic coil;
 n_B –number of turns per unit length;
 R_I –magnetic coil radius;

x_B –distance from the flexible ring to the magnetic coil plane;

θ –position angle of the flexible ring.

Equation (1) can be simplified to

$$q_{re} = q_{r0} \cos \theta \quad (2)$$

where $q_{r0} = \frac{n_B^2 \mu_0 R_I^4 l}{8(R_I^2 + x_B^2)^3} I^2$

The van der Waals force per unit length is [27]

$$q_{r3} = \frac{Al}{6\pi(t_0 - u_0)^3} \quad (3)$$

Where A is Hamaker constant, $A = 6.58 \times 10^{-20} J$.

The total force per unit length on the flexible ring is

$$q_{rs} = q_{re} + q_{r3} \quad (4)$$

The distribution of forces on the flexible wheel is as follows:

See equation (5) below.

This load is expanded as a Fourier series

$$q_{rs} = q_0 + \sum_1^{\infty} q_k \cos k\theta \quad (6)$$

The coefficients of the series satisfy the following conditions

See equation (7) below.

Thus

$$q_0 = \frac{2q_{r0}}{\pi} + \frac{q_{r3}}{2} \quad (8)$$

To determine the coefficient q_k , multiply the above integral equation by $\cos k\theta$, the following relation is obtained

See equation (9) below.

$$\begin{cases} q_{rs} = q_{r0} \cos \theta + q_{r3} = \frac{n_B^2 \mu_0 R_I^4 l}{8(R_I^2 + x_B^2)^3} I^2 \cos \theta + q_{r3} & \left(-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}\right) \\ q_{rs} = q_{r0} \cos \theta & \left(\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}, -\frac{\pi}{2} \leq \theta \leq -\frac{\pi}{4}\right) \\ q_{rs} = -q_{r0} \cos \theta + q_{r3} = -\frac{n_B^2 \mu_0 R_I^4 l}{8(R_I^2 + x_B^2)^3} I^2 \cos \theta + q_{r3} & \left(\frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}\right) \\ q_{rs} = -q_{r0} \cos \theta & \left(\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}, \frac{4\pi}{4} \leq \theta \leq \frac{3\pi}{2}\right) \end{cases} \quad (5)$$

$$\begin{aligned} \int_0^{2\pi} \left[q_0 + \sum_1^{\infty} q_k \cos k\theta \right] d\theta &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (q_{r0} \cos \theta + q_{r3}) d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (q_{r0} \cos \theta) d\theta \\ &+ \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} (-q_{r0} \cos \theta + q_{r3}) d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} (-q_{r0} \cos \theta) d\theta \\ &+ \int_{\frac{5\pi}{4}}^{\frac{3\pi}{2}} (-q_{r0} \cos \theta) d\theta + \int_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} (q_{r0} \cos \theta) d\theta \end{aligned} \quad (7)$$

$$\begin{aligned} \int_0^{2\pi} \left[q_0 + \sum_1^{\infty} q_k \cos k\theta \right] \cos k\theta d\theta &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (q_{r0} \cos \theta + q_{r3}) \cos k\theta d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (q_{r0} \cos \theta) \cos k\theta d\theta \\ &+ \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} (-q_{r0} \cos \theta + q_{r3}) \cos k\theta d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} (-q_{r0} \cos \theta) \cos k\theta d\theta \\ &+ \int_{\frac{5\pi}{4}}^{\frac{3\pi}{2}} (-q_{r0} \cos \theta) \cos k\theta d\theta + \int_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} (q_{r0} \cos \theta) \cos k\theta d\theta \end{aligned} \quad (9)$$

In order to make the load symmetric with respect to the vertical axis and the horizontal axis, only take the coefficient of the cosine series to be even, yields

$$q_k = \frac{2q_{r0}}{\pi} \left[\frac{\sin \frac{(k+1)\pi}{2}}{k+1} + \frac{\sin \frac{(k-1)\pi}{2}}{k-1} \right] + \frac{4q_{r3}}{\pi k} \sin \frac{k\pi}{4} \quad (k \neq 1) \quad (10)$$

Equation (10) can be changed into following form

$$q_{rs} = \frac{2q_{r0}}{\pi} + \frac{q_{r3}}{2} + \sum_{k=2,4,6,\dots} \left\{ \frac{2q_{r0}}{\pi} \left[\frac{\sin \frac{(k+1)\pi}{2}}{k+1} + \frac{\sin \frac{(k-1)\pi}{2}}{k-1} \right] + \frac{4q_{r3}}{\pi k} \sin \frac{k\pi}{4} \right\} \cos k\theta \quad (11)$$

Substituting q_{rs} into the dynamics equation of the flexible wheel:

$$\frac{\partial^5 u}{\partial \theta^5} + 2 \frac{\partial^3 u}{\partial \theta^3} + \frac{\partial u}{\partial \theta} = \frac{r^4}{EI_x} \frac{\partial q_{rs}}{\partial \theta} - \frac{r^4 \rho A}{EI_x} \frac{\partial \ddot{u}}{\partial \theta} \quad (12)$$

This is a static solution, so the derivative of the displacement with respect to time is zero, that is, $\partial \ddot{u} / \partial \theta = 0$, yields

$$\frac{\partial^5 u}{\partial \theta^5} + 2 \frac{\partial^3 u}{\partial \theta^3} + \frac{\partial u}{\partial \theta} = - \frac{r^4}{EI_x \pi} \sum_{k=2,4,6,\dots} \left\{ 2q_{r0} \left[\frac{k \sin \frac{(k+1)\pi}{2}}{k+1} + \frac{k \sin \frac{(k-1)\pi}{2}}{k-1} \right] + 4q_{r3} \sin \frac{k\pi}{4} \right\} \sin k\theta \quad (13)$$

Write the solution of equation (13) as a series as follows

$$u = \sum_{k=2,4,6,\dots} C_k \cos k\theta \quad (14)$$

Substituting equation (14) into equation (13) to obtain

$$\sum_{k=2,4,6,\dots} C_k (-k^5 + 2k^3 - k) \sin k\theta = - \frac{r^4}{EI_x} \sum_{k=2,4,6,\dots} \left\{ \frac{2q_{r0}k}{\pi} \left[\frac{\sin \frac{(k+1)\pi}{2}}{k+1} + \frac{\sin \frac{(k-1)\pi}{2}}{k-1} \right] + \frac{4q_{r3}}{\pi} \sin \frac{k\pi}{4} \right\} \sin k\theta \quad (15)$$

On both sides of equation (15), if the corresponding terms of the same k value are equal, then

$$C_k = \frac{r^4}{EI_x} \frac{\frac{2q_{r0}k}{\pi} \left[\frac{\sin \frac{(k+1)\pi}{2}}{k+1} + \frac{\sin \frac{(k-1)\pi}{2}}{k-1} \right] + \frac{4q_{r3}}{\pi} \sin \frac{k\pi}{4}}{k(k^2 - 1)^2} \quad (16)$$

So

See equation (17) below.

Letting u_0 be equal to the average value of the displacement u at the angle range $[-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}]$, using equation (17), u_0 can be obtained

See equation (18) below.

$$u = \frac{r^4}{\pi EI_x} \sum_{k=2,4,6,\dots} \left[2q_{r0} \frac{(k-1) \sin \frac{(k+1)\pi}{2} + (k+1) \sin \frac{(k-1)\pi}{2}}{(k^2 - 1)^3} + 4q_{r3} \frac{\sin \frac{k\pi}{4}}{k(k^2 - 1)^2} \right] \cos k\theta \quad (17)$$

$$\begin{aligned} u_0 &= \frac{2}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} u d\theta \\ &= \frac{8r^4}{\pi^2 EI_x} \sum_{k=2,4,6,\dots} \left[q_{r0} \frac{(k-1) \sin \frac{(k+1)\pi}{2} + (k+1) \sin \frac{(k-1)\pi}{2}}{k(k^2 - 1)^3} + 2q_{r3} \frac{\sin \frac{k\pi}{4}}{k^2(k^2 - 1)^2} \right] \sin \frac{k\pi}{4} \end{aligned} \quad (18)$$

4 Solution of the dynamics equation for flexible ring

4.1 Mode function

Let the radial displacement u of the flexible ring be composed of static displacement u_0 and dynamic displacement Δu

$$u = u_0 + \Delta u \quad (19)$$

The radial load q_r is composed of static load q_{rs} and dynamic load Δq_r

$$q_r = q_{rs} + \Delta q_r \quad (20)$$

Substituting equation (20) into equation (12), yields

$$\frac{\partial^5 \Delta u}{\partial \theta^5} + 2 \frac{\partial^3 \Delta u}{\partial \theta^3} + \frac{\partial \Delta u}{\partial \theta} = \frac{r^4}{EI_x} \frac{\partial \Delta q_r}{\partial \theta} - \frac{r^4 \rho A_s}{EI_x} \frac{\partial \Delta \ddot{u}}{\partial \theta} \quad (21)$$

where $\Delta \ddot{u}$ is the second derivative of the dynamic displacement Δu with respect to time, E is the elastic modulus of the flexible ring material, ρ is density of the flexible ring, A_s is the section area of the flexible ring, I_x is the cross section modulus of the flexible ring.

Equation (21) can be changed into following form

$$\frac{\partial^4 \Delta u}{\partial \theta^4} + 2 \frac{\partial^2 \Delta u}{\partial \theta^2} + \Delta u = \frac{r^4}{EI_x} \Delta q_r - \frac{r^4 \rho A}{EI_x} \Delta \ddot{u} \quad (22)$$

where Δq_r is the dynamic load, $\Delta q_r = \frac{dq_r}{du} \Delta u$, from equations (1) and (3), the dynamic load can be determined by:

– Only considering electromagnetic force:

$$\Delta q_{re} = \frac{dq_{re}}{du} \Delta u_e = 0 \quad (23)$$

– Considering electromagnetic force and van der Waals force:

$$\begin{cases} \Delta q_{r3} = \frac{dq_{r3}}{du} \Delta u_3 = \frac{Al}{2\pi(t_0 - u_0)^4} \Delta u_3 & (0 \leq \theta \leq \frac{\pi}{4}) \\ \Delta q_{r3} = 0 & (\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}) \end{cases} \quad (24)$$

Substituting equations (23) and (24) into (22), the dynamics equation of the micro electromagnetic harmonic drive system is as follows:

– Only considering electromagnetic force:

$$\frac{\partial^4 \Delta u_e}{\partial \theta^4} + 2 \frac{\partial^2 \Delta u_e}{\partial \theta^2} + \Delta u_e = -\frac{r^4}{EI_x} \rho A_s \Delta \ddot{u}_e \quad (25)$$

– Considering electromagnetic force and van der Waals force:

See equation (26) below.

Letting

$$\Delta u = \phi(\theta) q(t) \quad (27)$$

Then

– Only considering electromagnetic force:

$$\frac{\ddot{q}(t)}{q(t)} = -\frac{\phi_e^{(4)}(\theta) + 2\phi_e''(\theta) + \phi_e(\theta)}{\frac{\rho A_s r^4}{EI_x} \phi_e(\theta)} \quad (28)$$

From equation (28), it is given

$$\ddot{q}(t) + \omega^2 q(t) = 0 \quad (29)$$

$$\phi_e^{(4)}(\theta) + 2\phi_e''(\theta) + Q\phi_e(\theta) = 0 \quad (30)$$

where $Q = 1 - \frac{\rho A_s r^4}{EI_x} \omega^2$

Letting $\phi_e(\theta) = e^{\lambda\theta}$ and substituting it into (30), yields

$$\lambda^4 + 2\lambda^2 + Q = 0 \quad (31)$$

The four eigen values can be obtained as follows $\pm \sqrt{-1 + \sqrt{1 - Q}}$ and $\pm i\sqrt{1 + \sqrt{1 - Q}}$, and then the mode function can be given by

$$\phi_e(\theta) = B_1 \cos k_1 \theta + B_2 \sin k_1 \theta + B_3 \cosh k_2 \theta + B_4 \sinh k_2 \theta \quad (32)$$

where $k_1 = \sqrt{1 + \sqrt{1 - Q}}$ and $k_2 = \sqrt{-1 + \sqrt{1 - Q}}$

The integral constants $B_j (j=1, 2, 3, 4)$ and frequency equation can be determined by symmetry and continuity conditions of the flexible ring.

– Considering electromagnetic force and van der Waals force:

(a) at $0 \leq \theta \leq \frac{\pi}{4}$

In a same manner, following equation can be given

$$\frac{\ddot{q}(t)}{q(t)} = -\frac{\phi_{31}^{(4)}(\theta) + 2\phi_{31}''(\theta) + P\phi_{31}(\theta)}{\frac{\rho A_s r^4}{EI_x} \phi_{31}(\theta)} \quad (33)$$

where $P = 1 - \frac{r^4 Al}{2EI_x \pi (t_0 - u_0)^4}$

$$\begin{cases} \frac{\partial^4 \Delta u_{13}}{\partial \theta^4} + 2 \frac{\partial^2 \Delta u_{13}}{\partial \theta^2} + \Delta u_{13} = \frac{r^4}{EI_x} \frac{Al}{2\pi(t_0 - u_0)^4} \Delta u_{13} - \frac{r^4}{EI_x} \rho A_s \Delta \ddot{u}_{13} & (0 \leq \theta \leq \frac{\pi}{4}) \\ \frac{\partial^4 \Delta u_{23}}{\partial \theta^4} + 2 \frac{\partial^2 \Delta u_{23}}{\partial \theta^2} + \Delta u_{23} = -\frac{r^4}{EI_x} \rho A_s \Delta \ddot{u}_{23} & (\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}) \end{cases} \quad (26)$$

From equation (33), it is obtained

$$\phi_{31}^{(4)}(\theta) + 2\phi_{31}''(\theta) + R\phi_{31}(\theta) = 0 \quad (34)$$

where $R = P - \frac{\rho A_s r^4}{EI_x} \omega^2$

Letting $\phi_{31}(\theta) = e^{\lambda\theta}$ and substituting it into (34), yields

$$\lambda^4 + 2\lambda^2 + R = 0 \quad (35)$$

Thus, the mode function can be given as

$$\phi_{31}(\theta) = A_1 \cos m_1 \theta + A_2 \sin m_1 \theta + A_3 \cosh m_2 \theta + A_4 \sinh m_2 \theta \quad (36)$$

where $m_1 = \sqrt{1 + \sqrt{1 - R}}$, $m_2 = \sqrt{-1 + \sqrt{1 - R}}$

(b) at $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

In a same manner, following equation can be given

$$\frac{\ddot{q}(t)}{q(t)} = - \frac{\phi_{32}^{(4)}(\theta) + 2\phi_{32}''(\theta) + \phi_{32}(\theta)}{\frac{\rho A_s r^4}{EI_x} \phi_{32}(\theta)} \quad (37)$$

From equation (37), it is obtained

$$\phi_{32}^{(4)}(\theta) + 2\phi_{32}''(\theta) + S\phi_{32}(\theta) = 0 \quad (38)$$

where $S = 1 - \frac{\rho A_s r^4}{EI_x} \omega^2$

Letting $\phi_{32}(\theta) = e^{\lambda\theta}$ and substituting it into (38), yields

$$\lambda^4 + 2\lambda^2 + S = 0 \quad (39)$$

Thus, the mode function can be given as

$$\phi_{32}(\theta) = A_5 \cos n_1 \theta + A_6 \sin n_1 \theta + A_7 \cosh n_2 \theta + A_8 \sinh n_2 \theta \quad (40)$$

where $n_1 = \sqrt{1 + \sqrt{1 - S}}$ and $n_2 = \sqrt{-1 + \sqrt{1 - S}}$.

The integral constants $A_j (j=1, 2, 3, 4, 5, 6, 7, 8)$ and frequency equation can be determined by symmetry and continuity conditions of the flexible ring.

4.2 Natural frequencies

From symmetry and continuity conditions of the flexible ring, both angle φ and tangent displacement v are zero at

$\theta = 0$ and $\theta = \frac{\pi}{2}$, so

$$\begin{cases} \phi'(\theta)_{\theta=0} = 0 \\ \phi'(\theta)_{\theta=\frac{\pi}{2}} = 0 \\ \phi^{(3)}(\theta)_{\theta=0} = 0 \\ \phi^{(3)}(\theta)_{\theta=\frac{\pi}{2}} = 0 \end{cases} \quad (41)$$

– Only considering electromagnetic force:

Substituting equation (41) into (32), yields

See equation (42) below.

From equation (42), it can be known that $B_2 = 0$ and $B_4 = 0$, so equation (42) can be changed into following form

$$C_e X_e = D_e \quad (43)$$

where

$$C_e = \begin{bmatrix} -k_1 \sin\left(\frac{\pi}{2} k_1\right) & k_2 \sinh\left(\frac{\pi}{2} k_2\right) \\ k_1^3 \sin\left(\frac{\pi}{2} k_1\right) & k_2^3 \sinh\left(\frac{\pi}{2} k_2\right) \end{bmatrix}$$

$$X_e = [B_1 \quad B_3]^T D_e = [0 \quad 0]^T$$

If there are non-zero solutions for equation (43), the determinant of is zero, that is

$$\begin{vmatrix} -k_1 \sin\left(\frac{\pi}{2} k_1\right) & k_2 \sinh\left(\frac{\pi}{2} k_2\right) \\ k_1^3 \sin\left(\frac{\pi}{2} k_1\right) & k_2^3 \sinh\left(\frac{\pi}{2} k_2\right) \end{vmatrix} = 0 \quad (44)$$

– Considering electromagnetic force and van der Waals force:

From continuity conditions of the flexible ring, following relationship can be given at $\theta = \frac{\pi}{4}$

$$\begin{cases} \phi_{31}(\theta) = \phi_{32}(\theta) \\ \phi_{31}'(\theta) = \phi_{32}'(\theta) \\ \phi_{31}''(\theta) = \phi_{32}''(\theta) \\ \phi_{31}^{(3)}(\theta) = \phi_{32}^{(3)}(\theta) \end{cases} \quad (45)$$

$$\left. \begin{aligned} B_2 k_1 + B_4 k_2 &= 0 \\ -B_1 k_1 \sin\left(\frac{\pi}{2} k_1\right) + B_2 k_1 \cos\left(\frac{\pi}{2} k_1\right) + B_3 k_2 \sinh\left(\frac{\pi}{2} k_2\right) + B_4 k_2 \cosh\left(\frac{\pi}{2} k_2\right) &= 0 \\ -B_2 k_1^3 + B_4 k_2^3 &= 0 \\ B_1 k_1^3 \sin\left(\frac{\pi}{2} k_1\right) - B_2 k_1^3 \cos\left(\frac{\pi}{2} k_1\right) + B_3 k_2^3 \sinh\left(\frac{\pi}{2} k_2\right) + B_4 k_2^3 \cosh\left(\frac{\pi}{2} k_2\right) &= 0 \end{aligned} \right\} \quad (42)$$

Substituting equations (36) and (40) into (45), yields following form

See equation (46) below.

$$C_1 X_1 = D_1 \quad (47)$$

where

From equation (46), it can be known that $A_2=0$ and $A_4=0$, so equation (46) can be changed into

See equation below.

$$\left. \begin{aligned} & A_2 m_1 + A_4 m_2 = 0 \\ & -A_5 n_1 \sin\left(\frac{\pi}{2} n_1\right) + A_6 n_1 \cos\left(\frac{\pi}{2} n_1\right) + A_7 n_2 sh\left(\frac{\pi}{2} n_2\right) + A_8 n_2 ch\left(\frac{\pi}{2} n_2\right) = 0 \\ & -A_2 m_1^3 + A_4 m_2^3 = 0 \\ & A_5 n_1^3 \sin\left(\frac{\pi}{2} n_1\right) - A_6 n_1^3 \cos\left(\frac{\pi}{2} n_1\right) + A_7 n_2^3 sh\left(\frac{\pi}{2} n_2\right) + A_8 n_2^3 ch\left(\frac{\pi}{2} n_2\right) = 0 \\ & A_1 \cos\left(\frac{\pi}{4} m_1\right) + A_2 \sin\left(\frac{\pi}{4} m_1\right) + A_3 ch\left(\frac{\pi}{4} m_2\right) + A_4 sh\left(\frac{\pi}{4} m_2\right) - A_5 \cos\left(\frac{\pi}{4} n_1\right) \\ & \quad - A_6 \sin\left(\frac{\pi}{4} n_1\right) - A_7 ch\left(\frac{\pi}{4} n_2\right) - A_8 sh\left(\frac{\pi}{4} n_2\right) = 0 \\ & -A_1 m_1 \sin\left(\frac{\pi}{4} m_1\right) + A_2 m_1 \cos\left(\frac{\pi}{4} m_1\right) + A_3 m_2 sh\left(\frac{\pi}{4} m_2\right) + A_4 m_2 ch\left(\frac{\pi}{4} m_2\right) \\ & \quad + A_5 n_1 \sin\left(\frac{\pi}{4} n_1\right) - A_6 n_1 \cos\left(\frac{\pi}{4} n_1\right) - A_7 n_2 sh\left(\frac{\pi}{4} n_2\right) - A_8 n_2 ch\left(\frac{\pi}{4} n_2\right) = 0 \\ & -A_1 m_1^2 \cos\left(\frac{\pi}{4} m_1\right) - A_2 m_1^2 \sin\left(\frac{\pi}{4} m_1\right) + A_3 m_2^2 ch\left(\frac{\pi}{4} m_2\right) + A_4 m_2^2 sh\left(\frac{\pi}{4} m_2\right) \\ & \quad + A_5 n_1^2 \cos\left(\frac{\pi}{4} n_1\right) + A_6 n_1^2 \sin\left(\frac{\pi}{4} n_1\right) - A_7 n_2^2 ch\left(\frac{\pi}{4} n_2\right) - A_8 n_2^2 sh\left(\frac{\pi}{4} n_2\right) = 0 \\ & A_1 m_1^3 \sin\left(\frac{\pi}{4} m_1\right) - A_2 m_1^3 \cos\left(\frac{\pi}{4} m_1\right) + A_3 m_2^3 sh\left(\frac{\pi}{4} m_2\right) + A_4 m_2^3 ch\left(\frac{\pi}{4} m_2\right) \\ & \quad - A_5 n_1^3 \sin\left(\frac{\pi}{4} n_1\right) + A_6 n_1^3 \cos\left(\frac{\pi}{4} n_1\right) - A_7 n_2^3 sh\left(\frac{\pi}{4} n_2\right) - A_8 n_2^3 ch\left(\frac{\pi}{4} n_2\right) = 0 \end{aligned} \right\} \quad (46)$$

$$C_1 = \begin{bmatrix} 0 & 0 & -n_1 \sin\left(\frac{\pi}{2} n_1\right) & n_1 \cos\left(\frac{\pi}{2} n_1\right) & n_2 sh\left(\frac{\pi}{2} n_2\right) & n_2 ch\left(\frac{\pi}{2} n_2\right) \\ 0 & 0 & n_1^3 \sin\left(\frac{\pi}{2} n_1\right) & -n_1^3 \cos\left(\frac{\pi}{2} n_1\right) & n_2^3 sh\left(\frac{\pi}{2} n_2\right) & n_2^3 ch\left(\frac{\pi}{2} n_2\right) \\ \cos\left(\frac{\pi}{4} m_1\right) & ch\left(\frac{\pi}{4} m_2\right) & -\cos\left(\frac{\pi}{4} n_1\right) & -\sin\left(\frac{\pi}{4} n_1\right) & -ch\left(\frac{\pi}{4} n_2\right) & -sh\left(\frac{\pi}{4} n_2\right) \\ -m_1 \sin\left(\frac{\pi}{4} m_1\right) & m_2 sh\left(\frac{\pi}{4} m_2\right) & n_1 \sin\left(\frac{\pi}{4} n_1\right) & -n_1 \cos\left(\frac{\pi}{4} n_1\right) & -n_2 sh\left(\frac{\pi}{4} n_2\right) & -n_2 ch\left(\frac{\pi}{4} n_2\right) \\ -m_1^2 \cos\left(\frac{\pi}{4} m_1\right) & m_2^2 ch\left(\frac{\pi}{4} m_2\right) & n_1^2 \cos\left(\frac{\pi}{4} n_1\right) & n_1^2 \sin\left(\frac{\pi}{4} n_1\right) & -n_2^2 ch\left(\frac{\pi}{4} n_2\right) & -n_2^2 sh\left(\frac{\pi}{4} n_2\right) \\ m_1^3 \sin\left(\frac{\pi}{4} m_1\right) & m_2^3 sh\left(\frac{\pi}{4} m_2\right) & -n_1^3 \sin\left(\frac{\pi}{4} n_1\right) & n_1^3 \cos\left(\frac{\pi}{4} n_1\right) & -n_2^3 sh\left(\frac{\pi}{4} n_2\right) & -n_2^3 ch\left(\frac{\pi}{4} n_2\right) \end{bmatrix}$$

$$X_1 = [A_1 \ A_3 \ A_5 \ A_6 \ A_7 \ A_8]^T D_1 = [0 \ 0 \ 0 \ 0 \ 0 \ 0]^T$$

If there are non-zero solutions for equation (47), the corresponding determinant is zero, that is

See equation (48) below.

Using equations (42) and (46), the constants B_j and A_j of mode functions can be determined. Using equation (44) and (48), natural frequencies of the flexible ring can be determined. Then, the primary mass M_{pi} of the first order can be calculated by

$$M_{pi} = \int_0^\beta \rho[\phi_{1i}(\theta)]^2 A r d\theta + \int_\beta^{\pi/2} \rho[\phi_{2i}(\theta)]^2 A r d\theta \quad (49)$$

Multiplying $\phi_i(\theta)$ by $M_{pi}^{-1/2}$, the normal mode function of the flexible ring is obtained.

5 Effects of friction and air damping

Under the electromagnetic force, the flexible wheel after deformation contacts with the outer ring stator at the angle range $[-\pi/4, \pi/4]$. Here, the friction force occurs. At other angle range, the air damping force is applied to the flexible wheel. Letting q_t denote friction force per unit length on the flexible wheel, q_p denote damping force per unit length on the flexible wheel.

From friction torque equation of the flexible wheel, the friction force q_{t3} considering electromagnetic force and Van der Waals force can be given by

$$\begin{aligned} q_{t3} &= \frac{\mu(q_{r0}Y_1 + 2q_{r3}Y_2)}{2Y_2} - \frac{\mu t_0 \pi E I_x}{4r^4 Y_2} \\ &= \frac{\mu q_{r0}Y_1}{2Y_2} - \frac{\mu t_0 \pi E I_x}{4r^4 Y_2} + \mu q_{r3} \end{aligned} \quad (50)$$

This friction force is expanded as a Fourier series and high order terms are neglected, the dynamic friction force can be obtained

$$\Delta q_{t3} = \frac{\mu A l}{2\pi(t_0 - u_0)^4} \Delta u \quad (51)$$

The air damping force q_p per unit length on the flexible wheel is [28]

$$q_p = \frac{\eta l^3}{(t_0 - u)^3} \frac{\partial u}{\partial t} \quad (52)$$

where η is the air dynamic viscosity, $\eta = 1.86 \times 10^{-5} \text{ N} \cdot \text{S} \cdot \text{m}^{-2}$.

This air damping force is expanded as a Fourier series and high order terms are neglected, the dynamic air damping force can be obtained

$$q_p = \frac{\eta l^3}{(t_0 - u_0)^3} \frac{\partial \Delta u}{\partial t} \quad (53)$$

Considering tangent friction force, the dynamics equation (12) of the flexible wheel is changed into following form

$$\frac{\partial^5 \Delta u}{\partial \theta^5} + 2 \frac{\partial^3 \Delta u}{\partial \theta^3} + \frac{\partial \Delta u}{\partial \theta} = \frac{r^4}{E I_x} \left(\frac{\partial \Delta q_r}{\partial \theta} + \Delta q_t \right) - \frac{r^4 \rho A_s}{E I_x} \frac{\partial \Delta \ddot{u}}{\partial \theta} \quad (54)$$

The forces on the flexible wheel include radial one and tangent one, and the force distribution can be given by

See equation (55) below.

$$\begin{vmatrix} 0 & 0 & -n_1 \sin\left(\frac{\pi}{2} n_1\right) & n_1 \cos\left(\frac{\pi}{2} n_1\right) & n_2 sh\left(\frac{\pi}{2} n_2\right) & n_2 ch\left(\frac{\pi}{2} n_2\right) \\ 0 & 0 & n_1^3 \sin\left(\frac{\pi}{2} n_1\right) & -n_1^3 \cos\left(\frac{\pi}{2} n_1\right) & n_2^3 sh\left(\frac{\pi}{2} n_2\right) & n_2^3 ch\left(\frac{\pi}{2} n_2\right) \\ \cos\left(\frac{\pi}{4} m_1\right) & ch\left(\frac{\pi}{4} m_2\right) & -\cos\left(\frac{\pi}{4} n_1\right) & -\sin\left(\frac{\pi}{4} n_1\right) & -ch\left(\frac{\pi}{4} n_2\right) & -sh\left(\frac{\pi}{4} n_2\right) \\ -m_1 \sin\left(\frac{\pi}{4} m_1\right) & m_2 sh\left(\frac{\pi}{4} m_2\right) & n_1 \sin\left(\frac{\pi}{4} n_1\right) & -n_1 \cos\left(\frac{\pi}{4} n_1\right) & -n_2 sh\left(\frac{\pi}{4} n_2\right) & -n_2 ch\left(\frac{\pi}{4} n_2\right) \\ -m_1^2 \cos\left(\frac{\pi}{4} m_1\right) & m_2^2 ch\left(\frac{\pi}{4} m_2\right) & n_1^2 \cos\left(\frac{\pi}{4} n_1\right) & n_1^2 \sin\left(\frac{\pi}{4} n_1\right) & -n_2^2 ch\left(\frac{\pi}{4} n_2\right) & -n_2^2 sh\left(\frac{\pi}{4} n_2\right) \\ m_1^3 \sin\left(\frac{\pi}{4} m_1\right) & m_2^3 sh\left(\frac{\pi}{4} m_2\right) & -n_1^3 \sin\left(\frac{\pi}{4} n_1\right) & n_1^3 \cos\left(\frac{\pi}{4} n_1\right) & -n_2^3 sh\left(\frac{\pi}{4} n_2\right) & -n_2^3 ch\left(\frac{\pi}{4} n_2\right) \end{vmatrix} = 0 \quad (48)$$

$$\begin{cases} \Delta q_{rp} = \frac{A l}{2\pi(t_0 - u_0)^4} \Delta u_p & \Delta q_t = \frac{\mu A l}{2\pi(t_0 - u_0)^4} \Delta u_p & \left(0 \leq \theta \leq \frac{\pi}{4}\right) \\ \Delta q_{rp} = -\frac{\eta l^3}{(t_0 - u_0)^3} \frac{\partial \Delta u_p}{\partial t} & \Delta q_t = 0 & \left(\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}\right) \end{cases} \quad (55)$$

Combining equation (54) with (55), yields

See equation (56) below.

Letting

$$\Delta u = \phi(\theta)q(t) \quad (57)$$

Substituting equation (57) into (56), yields:

See equation (58-a) and (58-b) below.

$$\text{where } T = \frac{r^4}{EI_x} \frac{Al}{2\pi(t_0 - u_0)^4}$$

From equation (58-a), we know

$$\phi_{p1}^{(5)}(\theta) + 2\phi_{p1}^{(3)}(\theta) + U\phi_{p1}'(\theta) - \mu T\phi_{p1}(\theta) = 0 \quad (59)$$

$$\text{where } U = 1 - T - \frac{\rho A_s r^4}{EI_x} \omega^2$$

Letting $\phi_{p1}(\theta) = e^{\lambda\theta}$ and substituting it into (59), yields

$$\lambda^5 + 2\lambda^3 + U\lambda - \mu T = 0 \quad (60)$$

The five eigenvalues can be obtained ($r_1, r_2 \pm ir_3$ and $r_4 \pm ir_5$), and then the mode function can be given by

$$\begin{aligned} \phi_{p1}(\theta) = & C_1 e^{r_1\theta} + e^{r_2\theta}(C_2 \cos r_3\theta + C_3 \sin r_3\theta) \\ & + e^{r_4\theta}(C_4 \cos r_5\theta + C_5 \sin r_5\theta) \end{aligned} \quad (61)$$

From equation (58-b), we know

$$\phi_{p2}^{(4)}(\theta) + 2\phi_{p2}''(\theta) + V\phi_{p2}(\theta) = 0 \quad (62)$$

$$\text{where } V = 1 - \frac{\rho A_s r^4}{EI_x} \omega^2$$

Letting $\phi_{p2}(\theta) = e^{\lambda\theta}$ and substituting it into (62), yields

$$\lambda^4 + 2\lambda^2 + V = 0 \quad (63)$$

The four eigenvalues can be obtained ($\pm\sqrt{-1+\sqrt{1-V}}$ and $\pm i\sqrt{1+\sqrt{1-V}}$), and then the mode function can be given by

$$\phi_{p2}(\theta) = C_6 \cos \alpha_1 \theta + C_7 \sin \alpha_1 \theta + C_8 \cosh \alpha_2 \theta + C_9 \sinh \alpha_2 \theta \quad (64)$$

where $\alpha_1 = \sqrt{1+\sqrt{1-V}}$ and $\alpha_2 = \sqrt{-1+\sqrt{1-V}}$

From symmetry and continuity conditions of the flexible ring, both angle φ , tangent displacement v and shear force are zero at $\theta = 0$ and $\theta = \frac{\pi}{2}$, so

$$\left\{ \begin{array}{l} \phi_{p1}'(\theta)_{\theta=0} = 0 \\ \phi_{p2}'(\theta)_{\theta=\frac{\pi}{2}} = 0 \\ \phi_{p1}^{(3)}(\theta)_{\theta=0} = 0 \\ \phi_{p2}^{(3)}(\theta)_{\theta=\frac{\pi}{2}} = 0 \end{array} \right. \quad (65)$$

From continuity conditions of the flexible ring, following relationship can be given at $\theta = \frac{\pi}{4}$

$$\left\{ \begin{array}{l} \phi_{p1}(\theta) = \phi_{p2}(\theta) \\ \phi_{p1}'(\theta) = \phi_{p2}'(\theta) \\ \phi_{p1}''(\theta) = \phi_{p2}''(\theta) \\ \phi_{p1}^{(3)}(\theta) = \phi_{p2}^{(3)}(\theta) \\ \phi_{p1}^{(4)}(\theta) = \phi_{p2}^{(4)}(\theta) \end{array} \right. \quad (66)$$

$$\left\{ \begin{array}{l} \frac{\partial^5 \Delta u_{1p}}{\partial \theta^5} + 2 \frac{\partial^3 \Delta u_{1p}}{\partial \theta^3} + \frac{\partial \Delta u_{1p}}{\partial \theta} \\ = \frac{r^4}{EI_x} \left(\frac{Al}{2\pi(t_0 - u_0)^4} \frac{\partial \Delta u_{1p}}{\partial \theta} + \frac{\mu Al}{2\pi(t_0 - u_0)^4} \Delta u_{1p} \right) - \frac{r^4 \rho A_s}{EI_x} \frac{\partial \Delta \ddot{u}_{1p}}{\partial \theta} \quad \left(0 \leq \theta \leq \frac{\pi}{4} \right) \\ \frac{\partial^4 \Delta u_{2p}}{\partial \theta^4} + 2 \frac{\partial^2 \Delta u_{2p}}{\partial \theta^2} + \Delta u_{2p} = - \frac{r^4}{EI_x} \frac{\eta l^3}{(t_0 - u_0)^3} \frac{\partial \Delta u_{2p}}{\partial t} - \frac{r^4}{EI_x} \rho A_s \Delta \ddot{u}_{2p} \quad \left(\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2} \right) \end{array} \right. \quad (56)$$

$$\frac{\ddot{q}(t)}{q(t)} = - \frac{\phi_{p1}^{(5)}(\theta) + 2\phi_{p1}^{(3)}(\theta) + (1-T)\phi_{p1}'(\theta) - \mu T\phi_{p1}(\theta)}{\frac{\rho A_s r^4}{EI_x} \phi_{p1}'(\theta)} \quad \left(0 \leq \theta \leq \frac{\pi}{4} \right) \quad (58-a)$$

$$\frac{\ddot{q}(t)}{q(t)} + \frac{\eta l^3}{(t_0 - u_0)^3 \rho A_s q(t)} = - \frac{\phi_{p2}^{(4)}(\theta) + 2\phi_{p2}''(\theta) + \phi_{p2}(\theta)}{\frac{\rho A_s r^4}{EI_x} \phi_{p2}(\theta)} \quad \left(\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2} \right) \quad (58-b)$$

Substituting equations (61) and (64) into (65) and (66), yields

Equation (67) can be changed into following form

$$\text{See equation (67) below.} \quad CX_1 = D_1 \quad (68)$$

$$\left. \begin{aligned} & C_1 r_1 + C_2 r_2 + C_3 r_3 + C_4 r_4 + C_5 r_5 = 0 \\ & -C_6 \alpha_1 \sin\left(\frac{\pi}{2} \alpha_1\right) + C_7 \alpha_1 \cos\left(\frac{\pi}{2} \alpha_1\right) + C_8 \alpha_2 sh\left(\frac{\pi}{2} \alpha_2\right) + C_9 \alpha_2 ch\left(\frac{\pi}{2} \alpha_2\right) = 0 \\ & C_1 r_1^3 + C_2 (r_2^3 - 3r_2 r_3^2) + C_3 (3r_2^2 r_3 - r_3^3) + C_4 (r_4^3 - 3r_4 r_5^2) + C_5 (3r_4^2 r_5 - r_5^3) = 0 \\ & C_6 \alpha_1^3 \sin\left(\frac{\pi}{2} \alpha_1\right) - C_7 \alpha_1^3 \cos\left(\frac{\pi}{2} \alpha_1\right) + C_8 \alpha_2^3 sh\left(\frac{\pi}{2} \alpha_2\right) + C_9 \alpha_2^3 ch\left(\frac{\pi}{2} \alpha_2\right) = 0 \\ & C_1 e^{\frac{\pi}{4} r_1} + C_2 e^{\frac{\pi}{4} r_2} \cos\left(\frac{\pi}{4} r_3\right) + C_3 e^{\frac{\pi}{4} r_2} \sin\left(\frac{\pi}{4} r_3\right) + C_4 e^{\frac{\pi}{4} r_4} \cos\left(\frac{\pi}{4} r_5\right) + C_5 e^{\frac{\pi}{4} r_4} \sin\left(\frac{\pi}{4} r_5\right) \\ & -C_6 \cos\left(\frac{\pi}{4} \alpha_1\right) - C_7 \sin\left(\frac{\pi}{4} \alpha_1\right) - C_8 ch\left(\frac{\pi}{4} \alpha_2\right) - C_9 sh\left(\frac{\pi}{4} \alpha_2\right) = 0 \\ & C_1 r_1 e^{\frac{\pi}{4} r_1} + C_2 e^{\frac{\pi}{4} r_2} \left(r_2 \cos\left(\frac{\pi}{4} r_3\right) - r_3 \sin\left(\frac{\pi}{4} r_3\right)\right) + C_3 e^{\frac{\pi}{4} r_2} \left(r_2 \sin\left(\frac{\pi}{4} r_3\right) + r_3 \cos\left(\frac{\pi}{4} r_3\right)\right) \\ & + C_4 e^{\frac{\pi}{4} r_4} \left(r_4 \cos\left(\frac{\pi}{4} r_5\right) - r_5 \sin\left(\frac{\pi}{4} r_5\right)\right) + C_5 e^{\frac{\pi}{4} r_4} \left(r_4 \sin\left(\frac{\pi}{4} r_5\right) + r_5 \cos\left(\frac{\pi}{4} r_5\right)\right) \\ & + C_6 \alpha_1 \sin\left(\frac{\pi}{4} \alpha_1\right) - C_7 \alpha_1 \cos\left(\frac{\pi}{4} \alpha_1\right) - C_8 \alpha_2 sh\left(\frac{\pi}{4} \alpha_2\right) - C_9 \alpha_2 ch\left(\frac{\pi}{4} \alpha_2\right) = 0 \\ & C_1 r_1^2 e^{\frac{\pi}{4} r_1} + C_2 e^{\frac{\pi}{4} r_2} \left((r_2^2 - r_3^2) \cos\left(\frac{\pi}{4} r_3\right) - 2r_2 r_3 \sin\left(\frac{\pi}{4} r_3\right)\right) \\ & + C_3 e^{\frac{\pi}{4} r_2} \left((r_2^2 - r_3^2) \sin\left(\frac{\pi}{4} r_3\right) + 2r_2 r_3 \cos\left(\frac{\pi}{4} r_3\right)\right) \\ & + C_4 e^{\frac{\pi}{4} r_4} \left((r_4^2 - r_5^2) \cos\left(\frac{\pi}{4} r_5\right) - 2r_4 r_5 \sin\left(\frac{\pi}{4} r_5\right)\right) \\ & + C_5 e^{\frac{\pi}{4} r_4} \left((r_4^2 - r_5^2) \sin\left(\frac{\pi}{4} r_5\right) + 2r_4 r_5 \cos\left(\frac{\pi}{4} r_5\right)\right) \\ & + C_6 \alpha_1^2 \cos\left(\frac{\pi}{4} \alpha_1\right) + C_7 \alpha_1^2 \sin\left(\frac{\pi}{4} \alpha_1\right) - C_8 \alpha_2^2 ch\left(\frac{\pi}{4} \alpha_2\right) - C_9 \alpha_2^2 sh\left(\frac{\pi}{4} \alpha_2\right) = 0 \\ & C_1 r_1^3 e^{\frac{\pi}{4} r_1} + C_2 e^{\frac{\pi}{4} r_2} \left((r_3^3 - 3r_2^2 r_3) \sin\left(\frac{\pi}{4} r_3\right) + (r_2^3 - 3r_2 r_3^2) \cos\left(\frac{\pi}{4} r_3\right)\right) \\ & + C_3 e^{\frac{\pi}{4} r_2} \left((r_2^3 - 3r_2 r_3^2) \sin\left(\frac{\pi}{4} r_3\right) + (3r_2^2 r_3 - r_3^3) \cos\left(\frac{\pi}{4} r_3\right)\right) \\ & + C_4 e^{\frac{\pi}{4} r_4} \left((r_5^3 - 3r_4^2 r_5) \sin\left(\frac{\pi}{4} r_5\right) + (r_4^3 - 3r_4 r_5^2) \cos\left(\frac{\pi}{4} r_5\right)\right) \\ & + C_5 e^{\frac{\pi}{4} r_4} \left((r_4^3 - 3r_4 r_5^2) \sin\left(\frac{\pi}{4} r_5\right) + (3r_4^2 r_5 - r_5^3) \cos\left(\frac{\pi}{4} r_5\right)\right) \\ & -C_6 \alpha_1^3 \sin\left(\frac{\pi}{4} \alpha_1\right) + C_7 \alpha_1^3 \cos\left(\frac{\pi}{4} \alpha_1\right) - C_8 \alpha_2^3 sh\left(\frac{\pi}{4} \alpha_2\right) - C_9 \alpha_2^3 ch\left(\frac{\pi}{4} \alpha_2\right) = 0 \\ & C_1 r_1^4 e^{\frac{\pi}{4} r_1} + C_2 e^{\frac{\pi}{4} r_2} \left((4r_2 r_3^3 - 4r_2^3 r_3) \sin\left(\frac{\pi}{4} r_3\right) + (r_2^4 - 6r_2^2 r_3^2 + r_3^4) \cos\left(\frac{\pi}{4} r_3\right)\right) \\ & + C_3 e^{\frac{\pi}{4} r_2} \left((r_2^4 - 6r_2^2 r_3^2 + r_3^4) \sin\left(\frac{\pi}{4} r_3\right) + (4r_2^3 r_3 - 4r_2 r_3^3) \cos\left(\frac{\pi}{4} r_3\right)\right) \\ & + C_4 e^{\frac{\pi}{4} r_4} \left((4r_4 r_5^3 - 4r_4^3 r_5) \sin\left(\frac{\pi}{4} r_5\right) + (r_4^4 - 6r_4^2 r_5^2 + r_5^4) \cos\left(\frac{\pi}{4} r_5\right)\right) \\ & + C_5 e^{\frac{\pi}{4} r_4} \left((r_4^4 - 6r_4^2 r_5^2 + r_5^4) \sin\left(\frac{\pi}{4} r_5\right) + (4r_4^3 r_5 - 4r_4 r_5^3) \cos\left(\frac{\pi}{4} r_5\right)\right) \\ & -C_6 \alpha_1^4 \cos\left(\frac{\pi}{4} \alpha_1\right) - C_7 \alpha_1^4 \sin\left(\frac{\pi}{4} \alpha_1\right) - C_8 \alpha_2^4 sh\left(\frac{\pi}{4} \alpha_2\right) - C_9 \alpha_2^4 ch\left(\frac{\pi}{4} \alpha_2\right) = 0 \end{aligned} \right\} \quad (67)$$

where

$$C = \begin{bmatrix} r_1 & r_2 & r_3 & r_4 & r_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_1 \sin\left(\frac{\pi}{2}\alpha_1\right) & \alpha_1 \cos\left(\frac{\pi}{2}\alpha_1\right) & \alpha_2 sh\left(\frac{\pi}{2}\alpha_2\right) & \alpha_2 ch\left(\frac{\pi}{2}\alpha_2\right) \\ r_1^3 & H_{11} & H_{12} & H_{13} & H_{14} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_1^3 \sin\left(\frac{\pi}{2}\alpha_1\right) & -\alpha_1^3 \cos\left(\frac{\pi}{2}\alpha_1\right) & \alpha_2^3 sh\left(\frac{\pi}{2}\alpha_2\right) & \alpha_2^3 ch\left(\frac{\pi}{2}\alpha_2\right) \\ \frac{\pi}{e^4}r_1 & H_{21} & H_{22} & H_{23} & H_{24} & -\cos\left(\frac{\pi}{4}\alpha_1\right) & -\sin\left(\frac{\pi}{4}\alpha_1\right) & -ch\left(\frac{\pi}{4}\alpha_2\right) & -sh\left(\frac{\pi}{4}\alpha_2\right) \\ r_1 \frac{\pi}{e^4}r_1 & H_{31} & H_{32} & H_{33} & H_{34} & \alpha_1 \sin\left(\frac{\pi}{4}\alpha_1\right) & -\alpha_1 \cos\left(\frac{\pi}{4}\alpha_1\right) & -\alpha_2 sh\left(\frac{\pi}{4}\alpha_2\right) & -\alpha_2 ch\left(\frac{\pi}{4}\alpha_2\right) \\ r_1^2 \frac{\pi}{e^4}r_1 & H_{41} & H_{42} & H_{43} & H_{44} & \alpha_1^2 \cos\left(\frac{\pi}{4}\alpha_1\right) & \alpha_1^2 \sin\left(\frac{\pi}{4}\alpha_1\right) & -\alpha_2^2 ch\left(\frac{\pi}{4}\alpha_2\right) & -\alpha_2^2 sh\left(\frac{\pi}{4}\alpha_2\right) \\ r_1^3 \frac{\pi}{e^4}r_1 & H_{51} & H_{52} & H_{53} & H_{54} & -\alpha_1^3 \sin\left(\frac{\pi}{4}\alpha_1\right) & \alpha_1^3 \cos\left(\frac{\pi}{4}\alpha_1\right) & -\alpha_2^3 sh\left(\frac{\pi}{4}\alpha_2\right) & -\alpha_2^3 ch\left(\frac{\pi}{4}\alpha_2\right) \\ r_1^4 \frac{\pi}{e^4}r_1 & H_{61} & H_{62} & H_{63} & H_{64} & -\alpha_1^4 \cos\left(\frac{\pi}{4}\alpha_1\right) & -\alpha_1^4 \sin\left(\frac{\pi}{4}\alpha_1\right) & -\alpha_2^4 sh\left(\frac{\pi}{4}\alpha_2\right) & -\alpha_2^4 ch\left(\frac{\pi}{4}\alpha_2\right) \end{bmatrix}$$

here

$$H_{61} = e^{\frac{\pi}{4}r_2} \left((4r_2r_3^3 - 4r_2^3r_3) \sin\left(\frac{\pi}{4}r_3\right) + (r_2^4 - 6r_2^2r_3^2 + r_3^4) \cos\left(\frac{\pi}{4}r_3\right) \right),$$

$$H_{62} = e^{\frac{\pi}{4}r_2} \left((r_2^4 - 6r_2^2r_3^2 + r_3^4) \sin\left(\frac{\pi}{4}r_3\right) + (4r_2^3r_3 - 4r_2r_3^3) \cos\left(\frac{\pi}{4}r_3\right) \right),$$

$$H_{63} = e^{\frac{\pi}{4}r_4} \left((4r_4r_5^3 - 4r_4^3r_5) \sin\left(\frac{\pi}{4}r_5\right) + (r_4^4 - 6r_4^2r_5^2 + r_5^4) \cos\left(\frac{\pi}{4}r_5\right) \right),$$

$$H_{64} = e^{\frac{\pi}{4}r_4} \left((r_4^4 - 6r_4^2r_5^2 + r_5^4) \sin\left(\frac{\pi}{4}r_5\right) + (4r_4^3r_5 - 4r_4r_5^3) \cos\left(\frac{\pi}{4}r_5\right) \right),$$

$$H_{51} = e^{\frac{\pi}{4}r_2} \left((r_3^3 - 3r_2^2r_3) \sin\left(\frac{\pi}{4}r_3\right) + (r_2^3 - 3r_2r_3^2) \cos\left(\frac{\pi}{4}r_3\right) \right),$$

$$H_{52} = e^{\frac{\pi}{4}r_2} \left((r_2^3 - 3r_2r_3^2) \sin\left(\frac{\pi}{4}r_3\right) + (3r_2^2r_3 - r_3^3) \cos\left(\frac{\pi}{4}r_3\right) \right),$$

$$H_{53} = e^{\frac{\pi}{4}r_4} \left((r_5^3 - 3r_4^2r_5) \sin\left(\frac{\pi}{4}r_5\right) + (r_4^3 - 3r_4r_5^2) \cos\left(\frac{\pi}{4}r_5\right) \right),$$

$$H_{54} = e^{\frac{\pi}{4}r_4} \left((r_4^3 - 3r_4r_5^2) \sin\left(\frac{\pi}{4}r_5\right) + (3r_4^2r_5 - r_5^3) \cos\left(\frac{\pi}{4}r_5\right) \right),$$

$$H_{41} = e^{\frac{\pi}{4}r_2} \left((r_2^2 - r_3^2) \cos\left(\frac{\pi}{4}r_3\right) - 2r_2r_3 \sin\left(\frac{\pi}{4}r_3\right) \right),$$

$$H_{42} = e^{\frac{\pi}{4}r_2} \left((r_2^2 - r_3^2) \sin\left(\frac{\pi}{4}r_3\right) + 2r_2r_3 \cos\left(\frac{\pi}{4}r_3\right) \right),$$

$$H_{43} = e^{\frac{\pi}{4}r_4} \left((r_4^2 - r_5^2) \cos\left(\frac{\pi}{4}r_5\right) - 2r_4r_5 \sin\left(\frac{\pi}{4}r_5\right) \right),$$

$$H_{44} = e^{\frac{\pi}{4}r_4} \left((r_4^2 - r_5^2) \sin\left(\frac{\pi}{4}r_5\right) + 2r_4r_5 \cos\left(\frac{\pi}{4}r_5\right) \right),$$

$$H_{31} = e^{\frac{\pi}{4}r_2} \left(r_2 \cos\left(\frac{\pi}{4}r_3\right) - r_3 \sin\left(\frac{\pi}{4}r_3\right) \right),$$

$$H_{32} = e^{\frac{\pi}{4}r_2} \left(r_2 \sin\left(\frac{\pi}{4}r_3\right) + r_3 \cos\left(\frac{\pi}{4}r_3\right) \right),$$

$$H_{33} = e^{\frac{\pi}{4}r_4} \left(r_4 \cos\left(\frac{\pi}{4}r_5\right) - r_5 \sin\left(\frac{\pi}{4}r_5\right) \right),$$

$$H_{34} = e^{\frac{\pi}{4}r_4} \left(r_4 \sin\left(\frac{\pi}{4}r_5\right) + r_5 \cos\left(\frac{\pi}{4}r_5\right) \right),$$

$$H_{21} = e^{\frac{\pi}{4}r_2} \cos\left(\frac{\pi}{4}r_3\right), H_{22} = e^{\frac{\pi}{4}r_2} \sin\left(\frac{\pi}{4}r_3\right), H_{23} = e^{\frac{\pi}{4}r_4} \cos\left(\frac{\pi}{4}r_5\right),$$

$$H_{24} = e^{\frac{\pi}{4}r_4} \sin\left(\frac{\pi}{4}r_5\right),$$

$$H_{11} = r_2^3 - 3r_2r_3^2, H_{12} = 3r_2^2r_3 - r_3^3, H_{13} = r_4^3 - 3r_4r_5^2, H_{14} = 3r_4^2r_5 - r_5^3$$

$$X_1 = [C_1 \ C_2 \ C_3 \ C_4 \ C_5 \ C_6 \ C_7 \ C_8 \ C_9]^T,$$

$$D_1 = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T$$

If there are non-zero solutions for equation (68), the determinant is zero, that is

frequencies of the flexible ring can be determined. Substituting the natural frequencies into following equation

See equation (69) below.

$$\omega_p = \omega \sqrt{1 - \zeta^2} \quad (70)$$

Using equation (68), the constants C_j of mode functions can be determined. Using equation (69), natural

$$\text{where } \zeta = \frac{\eta l^3}{2\omega(t_0 - u_0)^3 \rho A_s}.$$

$$\begin{vmatrix} r_1 & r_2 & r_3 & r_4 & r_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_1 \sin\left(\frac{\pi}{2}\alpha_1\right) & \alpha_1 \cos\left(\frac{\pi}{2}\alpha_1\right) & \alpha_2 sh\left(\frac{\pi}{2}\alpha_2\right) & \alpha_2 ch\left(\frac{\pi}{2}\alpha_2\right) \\ r_1^3 & H_{11} & H_{12} & H_{13} & H_{14} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_1^3 \sin\left(\frac{\pi}{2}\alpha_1\right) & -\alpha_1^3 \cos\left(\frac{\pi}{2}\alpha_1\right) & \alpha_2^3 sh\left(\frac{\pi}{2}\alpha_2\right) & \alpha_2^3 ch\left(\frac{\pi}{2}\alpha_2\right) \\ \frac{\pi}{e^4}r_1 & H_{21} & H_{22} & H_{23} & H_{24} & -\cos\left(\frac{\pi}{4}\alpha_1\right) & -\sin\left(\frac{\pi}{4}\alpha_1\right) & -ch\left(\frac{\pi}{4}\alpha_2\right) & -sh\left(\frac{\pi}{4}\alpha_2\right) \\ \frac{\pi}{r_1 e^4}r_1 & H_{31} & H_{32} & H_{33} & H_{34} & \alpha_1 \sin\left(\frac{\pi}{4}\alpha_1\right) & -\alpha_1 \cos\left(\frac{\pi}{4}\alpha_1\right) & -\alpha_2 sh\left(\frac{\pi}{4}\alpha_2\right) & -\alpha_2 ch\left(\frac{\pi}{4}\alpha_2\right) \\ \frac{\pi}{r_1^2 e^4}r_1 & H_{41} & H_{42} & H_{43} & H_{44} & \alpha_1^2 \cos\left(\frac{\pi}{4}\alpha_1\right) & \alpha_1^2 \sin\left(\frac{\pi}{4}\alpha_1\right) & -\alpha_2^2 ch\left(\frac{\pi}{4}\alpha_2\right) & -\alpha_2^2 sh\left(\frac{\pi}{4}\alpha_2\right) \\ \frac{\pi}{r_1^3 e^4}r_1 & H_{51} & H_{52} & H_{53} & H_{54} & -\alpha_1^3 \sin\left(\frac{\pi}{4}\alpha_1\right) & \alpha_1^3 \cos\left(\frac{\pi}{4}\alpha_1\right) & -\alpha_2^3 sh\left(\frac{\pi}{4}\alpha_2\right) & -\alpha_2^3 ch\left(\frac{\pi}{4}\alpha_2\right) \\ \frac{\pi}{r_1^4 e^4}r_1 & H_{61} & H_{62} & H_{63} & H_{64} & -\alpha_1^4 \cos\left(\frac{\pi}{4}\alpha_1\right) & -\alpha_1^4 \sin\left(\frac{\pi}{4}\alpha_1\right) & -\alpha_2^4 sh\left(\frac{\pi}{4}\alpha_2\right) & -\alpha_2^4 ch\left(\frac{\pi}{4}\alpha_2\right) \end{vmatrix} = 0 \quad (69)$$

Table 1. Parameters of the system.

Parameters	Values	Parameters	Values
N	200	R_1 (mm)	10
x_B (mm)	10	t_0 (μm)	0.5
r (mm)	1.4	l (mm)	0.3
d (μm)	50	E (MPa)	95

Table 2. Changes of natural frequency with t_0 (rad/s).

$t_0/\mu\text{m}$		0.4	0.5	0.7	0.8	1	2
Order one	ω_e	6986.3	6986.3	6986.3	6986.3	6986.3	6986.3
	ω_3	2328.7	4206.9	5688.3	6081.9	6554	6956.6
	$\omega_e - \omega_3$	4657.6	2279.4	1298	904.4	432.3	29.7
Order two	ω_e	34931.3	34931.3	34931.3	34931.3	34931.3	34931.3
	ω_3	31041.3	33379.2	34532.5	34698	34835.9	34925.3
	$\omega_e - \omega_3$	3890	1552.1	398.8	233.3	95.4	6
Order three	ω_e	81506.3	81506.3	81506.3	81506.3	81506.3	81506.3
	ω_3	79937.4	80857.2	81336.5	81406.7	81465.5	81503.8
	$\omega_e - \omega_3$	1568.9	649.1	169.8	99.6	40.8	2.5
Order four	ω_e	146711.4	146711.4	146711.4	146711.4	146711.4	146711.4
	ω_3	145823.2	146348	146616.9	146656	146688.7	146710
	$\omega_e - \omega_3$	888.2	363.4	94.5	55.4	22.7	1.4

Table 3. Changes of natural frequency with r (rad/s).

r/mm		0.8	1	1.2	1.4	1.7
Order one	ω_e	21395.4	13693.1	9509.1	6986.3	4738.1
	ω_3	19300.8	11149	6854.8	4206.9	949.9
	$\omega_e - \omega_3$	2094.6	2544.1	2654.3	2279.4	3788.2
Order two	ω_e	106977.1	68465.3	47545.4	34931.3	23690.4
	ω_3	106478.2	67683.8	46414.4	33379.2	21352.4
	$\omega_e - \omega_3$	498.9	781.5	1131	1552.1	2338
Order three	ω_e	249613.1	159752.4	110939.2	81506.3	55277.6
	ω_3	249399.9	159419.5	110460.8	80857.2	54328.3
	$\omega_e - \omega_3$	213.2	332.9	478.4	649.1	949.3
Order four	ω_e	449303.7	287554.3	199690.5	146711.4	99499.8
	ω_3	449185.1	287369.1	199423.6	146348	98963.4
	$\omega_e - \omega_3$	118.6	185.2	266.9	363.4	536.4

From equation (70), the effects of the air damping on the natural frequencies can be given.

6 Results and discussion

Using equations given in this paper, the free vibration of the micro electromagnetic harmonic drive system is investigated. Parameters of the system are shown in Table 1.

Table 1 shows that the clearance between the flexible ring and stator is 500 nm and the effects of the Van der Waals force on the natural frequencies are quite significant and should be considered. Tables 2–4 give the first four orders of the natural frequencies and their changes along with system parameters. Results show:

- Considering molecule force, the natural frequencies of the flexible ring are decreased. With decreasing clearance

Table 4. Changes of natural frequency with d (rad/s).

$d/\mu\text{m}$		30	40	50	80	100	150
Order one	ω_e	4191.8	5589	6986.3	11178	13972.5	20958.8
	ω_3	954.1	1863	4206.9	9182.3	12418.9	20157.8
	$\omega_e - \omega_3$	3237.7	3726	2279.4	1995.7	1553.6	801
Order two	ω_e	20958.8	27945	34931.3	55890.1	69862.6	104793.9
	ω_3	16370.1	25477.4	33379.2	55291.8	69480.5	104624.3
	$\omega_e - \omega_3$	4588.7	2467.6	1552.1	598.3	382.1	169.6
Order three	ω_e	48903.8	65205.1	81506.3	130410.1	163012.7	244519
	ω_3	47146.6	64197.5	80857.2	130155.2	162849.4	244446.4
	$\omega_e - \omega_3$	1757.2	1007.6	649.1	254.9	163.3	72.6
Order four	ω_e	88026.8	117369.1	146711.4	234738.2	293422.8	440134.2
	ω_3	87014.4	116800.9	146348	234596.4	293332	440093.9
	$\omega_e - \omega_3$	1012.4	568.2	363.4	141.8	90.8	40.3

between the flexible ring and stator, the natural frequencies of the flexible ring are decreased more rapidly. This is due to the increased effects of electromagnetic force and molecular force on the flexible ring, resulting in the stiffness reduction of the coupling system.

- As clearance between the flexible ring and stator is relatively large ($t_0 > 1\mu\text{m}$), the decrease of the natural frequencies of the flexible ring is not obvious. At $t_0 = 1\mu\text{m}$, the relative error between the first order of the natural frequencies with and without considering molecule force is equal to $(\omega_e - \omega_3)/\omega_e = 6.2\%$. At $t_0 = 2\mu\text{m}$, the relative error is: $(\omega_e - \omega_3)/\omega_e = 0.4\%$. At $t_0 = 0.8\mu\text{m}$, the relative error between natural frequencies with and without considering molecule force is equal to: $(\omega_e - \omega_3)/\omega_e = 12.9\%$. At $t_0 = 0.5\mu\text{m}$ and $t_0 = 0.4\mu\text{m}$, the relative error between natural frequencies is equal to: $(\omega_e - \omega_3)/\omega_e = 32.6\%$. ($t_0 = 0.5\mu\text{m}$) and $t_0 = 0.4\mu\text{m}$, and $(\omega_e - \omega_3)/\omega_e = 66.7\%$. ($t_0 = 0.4\mu\text{m}$).
- As the order number of the vibration modes increases, effects of the molecule force on the natural frequencies of the flexible ring becomes weak as well. For example, at $t_0 = 0.4\mu\text{m}$, the relative error between the second order of the natural frequencies with and without considering molecule force is reduced to be 11.1%; the relative error between the third order of the natural frequencies is reduced to be 1.9%; and the relative error between the fourth order of the natural frequencies is reduced to be 0.6%.
- With increasing radius of the flexible ring, its natural frequencies are decreased. For a relatively large radius of the flexible ring, the relative error between the natural frequencies with and without considering molecule force becomes large. It shows that effects of the molecule force on the natural frequencies of the flexible ring increases with increasing radius of the flexible ring. As the order number of the vibration modes increases, effects of the molecule force on relationship between the natural frequencies and radius of the flexible ring becomes weak.

At radius $r = 1.4\text{mm}$ decrease of the natural frequencies caused by molecule force is 6.2% for mode one, 0.3% for mode two, 0.05% for mode three, and 0.02% for mode 4.

- With increasing thickness of the flexible ring, the natural frequencies of the flexible ring are increased. For a relatively large thickness of the flexible ring, the relative error between the natural frequencies with and without considering molecule force becomes small. It shows that effects of the molecule force on the natural frequencies of the flexible ring decreases with increasing thickness of the flexible ring. As the order number of the vibration modes increases, effects of the molecule force on relationship between the natural frequencies and thickness of the flexible ring becomes weak. At thickness $d = 40\text{mm}$, decrease of the natural frequencies caused by molecule force is 66.7% for mode one, 8.8% for mode two, 1.55% for mode three, and 0.48% for mode 4.

In a word, in dynamics performance design of the drive system, to determine its natural frequency accurately, the effects of the molecule force on the natural frequencies should be considered for smaller clearance between the flexible ring and stator, smaller thickness of the flexible ring and larger radius of the flexible ring.

Substituting above natural frequencies into equations (32), (36) and (40), the first four orders of the vibration modes can be obtained (see Fig. 3). It shows:

- As the molecule force is considered, for mode 1, at $\theta = 0$ and $\theta = \pi$, the amplitudes of the flexible ring vibrations decrease; at $\theta = \pi/2$ and $\theta = 3\pi/2$, the amplitudes of the flexible ring vibrations increase obviously. It is because there is larger molecule force near $\theta = 0$ and $\theta = \pi$ than other places.
- As the order number of the vibration modes increases, effects of the molecule force on the vibration modes of the flexible ring becomes weak as well. For mode 1, difference between modes with and without molecule force is 0.8376; for mode 2, 3 and 4, the difference between modes are 0.0464, 0.0168, and 0.0025, respectively.

Fig. 3. Differences between modes with and without molecule force. (a) Mode 1 (b) mode 2 (c) mode 3 (d) mode 4.

From mode analysis, it can be known that effects of the molecule force on the vibration modes are obvious only for modes 1 and 2. Here, for modes 1 and 2, effects of the molecule force on relationship between the vibration modes and other parameters are investigated (see Figs. 4–6). Results show:

- As the clearance between the flexible ring and stator is relatively small, effects of the molecule force on vibration amplitudes of the flexible ring become more obvious. At the clearance $t_0 = 0.4 \mu\text{m}$, the difference between the maximum vibration amplitudes for mode one with and without considering molecule force is 2.287. At the clearance $t_0 = 0.5 \mu\text{m}$, $t_0 = 1 \mu\text{m}$ and $t_0 = 2 \mu\text{m}$, the difference between the maximum vibration amplitudes is 1.700, 0.164 and 0.008, respectively. For mode two, the difference between the maximum vibration amplitudes is 0.136 for $t_0 = 0.4$, 0.047 for, 0.0002 for $t_0 = 2 \mu\text{m}$.
- As the radius r of the flexible ring increases (from 0.8 to 1.7 mm), vibration amplitudes of the flexible ring increase as well. Near $\theta = \pi/2$ and $\theta = 3\pi/2$, vibration amplitudes of the flexible ring increase are increased more obviously. Near $\theta = \pi/2$ and $\theta = 3\pi/2$, the difference between the maximum vibration amplitudes with and

without considering molecule force is large as well. For mode 1, the difference between the maximum vibration amplitudes with and without considering molecule force is 0.2861 at $r = 0.8 \text{ mm}$, 0.69 at $r = 1 \text{ mm}$, and 2.045 at $r = 1.7 \text{ mm}$. For mode 2, the difference between the maximum vibration amplitudes with and without considering molecule force is 0.005 at $r = 0.8 \text{ mm}$, 0.012 at $r = 1 \text{ mm}$, and 0.112 at $r = 1.7 \text{ mm}$.

The results show that effects of the molecule force on the maximum vibration amplitudes increases with increasing the radius r of the flexible ring.

- As the thickness d of the flexible ring increases (from $30 \mu\text{m}$ to $150 \mu\text{m}$), vibration amplitudes of the flexible ring decrease. As the thickness d of the flexible ring increases, effects of the molecule force on vibration amplitudes of the flexible ring become weak. For mode 1, the difference between the maximum vibration amplitudes with and without considering molecule force is 2.509 at $d = 30 \mu\text{m}$, 1.7 at $d = 50 \mu\text{m}$, and 0.09547 at $d = 150 \mu\text{m}$. For mode 2, the difference between the maximum vibration amplitudes with and without considering molecule force is 0.3066 at $d = 30 \mu\text{m}$, 0.04698 at $d = 150 \mu\text{m}$, and 0.0056 at $d = 150 \mu\text{m}$.

Fig. 4. Vibration modes for different clearance t_0 . (a) Mode 1 (b) $\Delta\phi = \phi_e - \phi_3$ for mode 1 (c) mode 2 (d) $\Delta\phi = \phi_e - \phi_3$ for mode 2.

It can be seen that effects of the molecule force on the maximum vibration amplitudes increases with decreasing the thickness d of the flexible ring and order number of the modes.

Using equations (69) and (70), the effects of the friction force and air damping on the natural frequencies are investigated. The parameters of the drive system are shown in Table 1. The effects of the friction force are given in Tables 5–8. The effects of the damping are given in Tables 9–12. Results show:

- When considering friction force, the natural frequencies of the flexible ring are decreased. With increasing order number of the modes, the effects of the friction force on the natural frequencies reduce rapidly and the effects can be neglected when the order number of the modes is above 2.
- When the clearance between the flexible ring and stator is decreased, the effects of the friction force on the natural frequencies become more significant. When the radius of the flexible ring is increased, the effects of the friction force on the natural frequencies decrease.
- When the thickness of the flexible ring increases, the frequency difference ($\omega_3 - \omega_t$) increases. This shows that the effects of the friction force on the natural frequencies become more significant for large thickness of the flexible ring. When the current in the coils increases, the frequency difference ($\omega_3 - \omega_t$) decreases. This shows that the effects of the friction force on the natural frequencies reduce for large coil current.
- When considering air damping force, the natural frequencies of the flexible ring are also decreased. With increasing order number of the modes, the effects of the air damping force on the natural frequencies reduce as well.
- When the clearance between the flexible ring and stator is decreased, the frequency difference ($\omega_t - \omega_p$) increases which shows that the effects of the air damping force on the natural frequencies increases with decreasing the clearance.
- When the radius of the flexible ring is increased, the frequency difference ($\omega_3 - \omega_t$) increases which shows that the effects of the air damping force on the natural frequencies increases with decreasing the radius.
- When the thickness of the flexible ring increases, the frequency difference ($\omega_3 - \omega_t$) decreases which shows that

Table 5. Changes of natural frequency with t_0 with and without friction force (rad/s).

$t_0/\mu\text{m}$		0.3	0.4	0.5	0.6	0.8	1
Order 1	ω_3	5711.9	6634.5	6848.1	6921	6966	6978.1
	ω_t	2328.7	2454.9	2372.4	2348.6	2334.8	2331.2
	$\omega_3 - \omega_t$	3383.2	4179.6	4475.7	4572.4	4631.2	4646.9
Order 2	ω_3	34544.2	34855.3	34902.9	34918.1	34927.2	34929.6
	ω_t	34544.2	34855.3	34902.9	34918.1	34927.2	34929.6
	$\omega_3 - \omega_t$	0	0	0	0	0	0
Order 3	ω_3	81341.5	81473.8	81494.2	81500.7	81504.6	81505.6
	ω_t	81341.5	81473.8	81494.2	81500.7	81504.6	81505.6
	$\omega_3 - \omega_t$	0	0	0	0	0	0

Table 6. Changes of natural frequency with r with and without friction force (rad/s).

r/mm		0.8	1	1.2	1.3	1.4	1.45
Order 1	ω_3	21069.1	13163.8	8693.7	7094.5	5711.9	4988.4
	ω_t	7233.2	4744.6	3522.8	3288.4	2328.7	2170.9
	$\omega_3 - \omega_t$	13835.9	8419.2	5170.9	3806.1	3383.2	2817.5
Order 2	ω_3	106910.4	68353.2	47356.4	40255.5	34544.2	32007.7
	ω_t	106910.4	68353.2	47356.4	40255.5	34544.2	32007.7
	$\omega_3 - \omega_t$	0	0	0	0	0	0
Order 2	ω_3	249584.6	159704.4	110858.4	94418.5	81341.5	75745.5
	ω_t	249584.6	159704.4	110858.4	94418.5	81341.5	75745.5
	$\omega_3 - \omega_t$	0	0	0	0	0	0

Table 7. Changes of natural frequency with d with and without friction force (rad/s).

$d/\mu\text{m}$		48	50	60	80	100	150
Order 1	ω_3	5200.3	5711.9	7605.4	10774.3	13721.5	20849.6
	ω_t	2235.6	2328.7	3147.1	3862.0	4736.2	7019.0
	$\omega_3 - \omega_t$	2964.7	3383.2	4458.3	6912.3	8985.3	13830.6
Order 2	ω_3	33002.9	34544.2	41734.4	55804.9	69811.1	104771.9
	ω_t	33002.9	34544.2	41734.4	55804.9	69811.1	104771.9
	$\omega_3 - \omega_t$	0	0	0	0	0	0
Order 3	ω_3	78020.4	81341.5	97729.3	130373.7	162990.6	244509.6
	ω_t	78020.4	81341.5	97729.3	130373.7	162990.6	244509.6
	$\omega_3 - \omega_t$	0	0	0	0	0	0

Table 8. Changes of natural frequency with I with and without friction force (rad/s).

I/A		0.05	0.1	0.13	0.15	0.16
Order 1	ω_3	5850.7	5711.9	5546.1	5347	5125.5
	ω_t	2328.7	2328.7	2328.7	2328.7	2328.7
	$\omega_3 - \omega_t$	3522.0	3383.2	3217.4	3018.3	2796.8
Order 2	ω_3	34608.2	34544.2	34456.1	34332.1	34171.1
	ω_t	34608.2	34544.2	34456.1	34332.1	34171.1
	$\omega_3 - \omega_t$	0	0	0	0	0
Order 3	ω_3	81368.6	81341.5	81304.2	81251.9	81184.4
	ω_t	81368.6	81341.5	81304.2	81251.9	81184.4
	$\omega_3 - \omega_t$	0	0	0	0	0

Table 9. Changes of natural frequency with t_0 with and without damping (rad/s).

$t_0/\mu\text{m}$		0.3	0.4	0.5	0.6	0.8	1
Order 1	ω_t	2328.7	2454.9	2372.4	2348.6	2334.8	2331.2
	ω_p	2312.4	2451.9	2371.2	2348.0	2334.6	2331.1
	$\omega_t - \omega_p$	16.3	3.0	1.2	0.6	0.2	0.1
Order 2	ω_t	34544.2	34855.3	34902.9	34918.1	34927.2	34929.6
	ω_p	34543.1	34855.1	34902.8	34918.1	34927.2	34929.6
	$\omega_t - \omega_p$	1.1	0.2	0.1	0	0	0
Order 3	ω_t	81341.5	81473.8	81494.2	81500.7	81504.6	81505.6
	ω_p	81341.0	81473.7	81494.2	81500.7	81504.6	81505.6
	$\omega_t - \omega_p$	0.5	0.1	0	0	0	0

Table 10. Changes of natural frequency with r with and without damping (rad/s).

r/mm		0.8	1	1.2	1.3	1.4	1.45
Order 1	ω_t	7233.2	4744.6	3522.8	3288.4	2328.7	2170.9
	ω_p	7230.4	4740.0	3515.6	3279.5	2312.4	2147.4
	$\omega_t - \omega_p$	2.8	4.6	7.2	8.9	16.3	23.5
Order 2	ω_t	106910.4	68353.2	47356.4	40255.5	34544.2	32007.7
	ω_p	106910.2	68352.9	47355.9	40254.8	34543.1	32006.1
	$\omega_t - \omega_p$	0.2	0.3	0.5	0.7	1.1	1.6
Order 3	ω_t	249584.6	159704.4	110858.4	94418.5	81341.5	75745.5
	ω_p	249584.5	159704.3	110858.2	94418.2	81341.0	75744.8
	$\omega_t - \omega_p$	0.1	0.1	0.2	0.3	0.5	0.7

Table 11. Changes of natural frequency with d with and without damping (rad/s).

$d/\mu\text{m}$		48	50	60	80	100	150
Order 1	ω_t	2235.6	2328.7	3147.1	3862.0	4736.2	7019.0
	ω_p	2212.2	2312.4	3141.4	3859.8	4735.1	7018.7
	$\omega_t - \omega_p$	23.4	16.3	5.7	2.2	1.1	0.3
Order 2	ω_t	33002.9	34544.2	41734.4	55804.9	69811.1	104771.9
	ω_p	33001.3	34543.1	41734.0	55804.7	69811.0	104771.9
	$\omega_t - \omega_p$	1.6	1.1	0.4	0.2	0.1	0
Order 3	ω_t	78020.4	81341.5	97729.3	130373.7	162990.6	244509.6
	ω_p	78019.7	81341.0	97729.1	130373.6	162990.6	244509.6
	$\omega_t - \omega_p$	0.7	0.5	0.2	0.1	0	0

Table 12. Changes of natural frequency with I with and without damping (rad/s).

I/A		0.05	0.1	0.13	0.15	0.16
Order 1	ω_t	2328.6	2328.7	2328.7	2328.7	2328.7
	ω_p	2315.0	2312.4	2308.6	2303.4	2296.6
	$\omega_t - \omega_p$	13.6	16.3	20.1	25.3	32.1
Order 2	ω_t	34608.2	34544.2	34456.1	34332.1	34171.1
	ω_p	34607.3	34543.1	34454.8	34330.4	34168.9
	$\omega_t - \omega_p$	0.9	1.1	1.3	1.7	2.2
Order 3	ω_t	81368.6	81341.5	81304.2	81251.9	81184.4
	ω_p	81368.2	81341.0	81303.6	81251.2	81183.5
	$\omega_t - \omega_p$	0.4	0.5	0.6	0.7	0.9

Fig. 5. Vibration modes for different radius r . (a) Mode 1 (b) $\Delta\phi = \phi_e - \phi_3$ for mode 1 (c) mode 2 (d) $\Delta\phi = \phi_e - \phi_3$ for mode 2.

the effects of the air damping force on the natural frequencies become weak for large thickness of the flexible ring.

- When the current in the coils increases, the frequency difference ($\omega_3 - \omega_t$) increases which shows that the effects of the air damping force on the natural frequencies increases for large coil current.

To illustrate the theoretical analysis, FEM software, ANSYS, is used to simulate the dynamics performance of the flexible ring. The simulating process is as follows:

- FEM model of the flexible ring is produced. The density, Young's modulus and Poisson's ratio of the flexible ring material are set.
- At $\theta = 0$ and $\theta = \pi/2$, the movement and rotation in other directions of the flexible ring are restricted, only the movement in radial direction is allowed.
- Calculated electromagnetic force and molecular force are converted into the force per unit area applied to the flexible ring. The electromagnetic force and molecular force are 0.0785 and 0.2156 N/m^2 , respectively. The direction of the forces is radial to the outside of the flexible wheel.

Table 13. Comparison of calculated and simulated natural frequencies.

Order number	Calculated (Hz)	Simulated (Hz)	Error (%)
1	909.077	1095.1	16.99
2	5497.88	5921.1	7.15
3	12945.9	13977	7.38
4	23335.23	25132	7.15

Using the FEM model, the natural frequencies of the flexible wheel are obtained and compared with the calculated ones (see Tab. 13). Table 13 show:

The calculated and simulated natural frequencies are in good agreement with each other. The relative error between them is below 17%. This illustrates theoretical analysis of the paper.

7 Conclusions

In this paper, considering Van der Waals force, dynamics equation of the flexible ring for the micro electromagnetic

Fig. 6. Vibration modes for different thickness d . (a) Mode 1 (b) $\Delta\phi = \phi_e - \phi_3$ for mode 1 (c) mode 2 (d) $\Delta\phi = \phi_e - \phi_3$ for mode 2.

harmonic drive system is proposed. Using the equations, effects of the molecule force on the natural frequencies and vibration modes of the drive system are investigated. Results show:

- Considering molecule force, natural frequencies of the flexible ring are reduced. For lower order modes, the effects of the molecule force on the natural frequencies are more obvious. For smaller clearance between the flexible ring and stator, smaller thickness of the flexible ring and larger radius of the flexible ring, the effects of the molecule force on the natural frequencies are more obvious.
- Considering molecule force, vibration modes of the flexible ring are changed. At some positions, the vibration amplitudes are decreased; at other positions, the vibration amplitudes are increased. For lower order modes, the effects of the molecule force on the vibration modes are more obvious. For smaller clearance between the flexible ring and stator, smaller thickness of the flexible ring and larger radius of the flexible ring, the effects of the molecule force on the vibration modes are more obvious.

- When considering friction force and air damping force, the natural frequencies of the flexible ring are decreased.

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